CS5314
Randomized Algorithms

Lecture 20: Probabilistic Method
(Lovasz Local Lemma)

Objectives

- Introduce Lovasz Local Lemma (LLL)
  - one of the most elegant and useful tools in the probabilistic method
- Two versions:
  - symmetric case
  - general case

Lovasz Local Lemma

- Let $E_1, E_2, \ldots, E_n$ be a set of BAD events
- Suppose each occurs with prob < 1

Fact: If they are mutually independent, it is easy to see that

$$\Pr(\text{no BAD events}) > 0 \quad \ldots \text{[why?]}$$

- However, in many natural scenario, the BAD events are not mutually independent

Problem: Can we still easily show that

$$\Pr(\text{no BAD events}) > 0\ ?$$

Lovasz Local Lemma (2)

- In general, probably not…
- But, if there are not many dependency among the BAD events, then the set of events are 'roughly' mutually independent

  ➔ we may still be able to show

  Pr(no BAD events) > 0 …

- Lovasz Local Lemma gives sufficient conditions when we can do so …
  - It relies on a concept of dependency graph defined as follows (next slide)

Dependency Graph

Let E be an event

Definition: E is mutually independent of a set of events $\{E_1, E_2, \ldots, E_n\}$ if

for any $I \subseteq [1,n]$, $\Pr(E \mid \bigcap_{j\in I} E_j) = \Pr(E)$

Definition: A dependency graph for a set of events $\{E_1, E_2, \ldots, E_n\}$ is a graph $G=(V,E)$, $V = \{1,2,\ldots,n\}$ such that

for any j, E_j is mutually independent of the events $\{ E_k \mid (j,k) \notin E \}$

Dependency Graph (2)

Test your understanding:

1. Let S be a set of pair-wise independent events. Is a graph with no edges always a dependency graph of S ?

2. Let S be a set of events.

   Is the dependency graph of S unique?

The answers are NO for both questions…

Dependency Graph (3)

Consider flipping a fair coin twice.

Let E_1 = the first flip is head

E_2 = the second flip is tail

E_3 = the two flips are the same

→ the events are pairwise independent

We see that if a graph has less than 2 edges, it must not be a dependency graph

On the other hand, any graph with 2 or more edges is a dependency graph !!!

Lovasz Local Lemma
(Symmetric Case)

Theorem: Let G be a dependency graph of a set of BAD events $\{ E_1, E_2, \ldots, E_n \}$. If

(i) $\Pr(E_j) \leq p < 1$ for each E_j,

(ii) $1 \leq \text{maxdeg}(G) \leq d$, and

(iii) $4pd \leq 1$

then $\Pr(\text{no BAD events}) > 0$

Remark: If $\text{maxdeg}(G) = 0$, then $\Pr(\text{no BAD events}) > 0$ since all events are mutually independent

Proof

Let $S = \{s_1, s_2, \ldots\}$ be a subset of $\{1, 2, \ldots, n\}$

- The proof is based on induction
- In particular, we show two statements are true alternately:

(1) $\Pr(E_k \mid \bigcap_{j \in S} \neg E_j) \leq 2p$ for all S,

with $|S| = 0, 1, 2, \ldots, n-1$

(2) $\Pr(\bigcap_{j \in S} \neg E_j) > 0$ for all S,

with $|S| = 1, 2, \ldots, n$

Proof (2)

- The base case(s) are : 1st statement with $|S|=0$, and 2nd statement with $|S|=1$
- For the inductive steps:

(1) Assume 1st statement is true for $|S| \leq h$ and 2nd statement is true for $|S| \leq h+1$

➔ prove 1st statement is true for $|S|=h+1$

(2) Assume 1st statement is true for $|S| \leq h+1$ and 2nd statement is true for $|S| \leq h+1$,

➔ prove 2nd statement is true for $|S|=h+2$

Proof (3)

Consequently, by induction,

we can prove the 1st statement when $|S|=1$,

and then the 2nd statement when $|S|=2$,

and then the 1st statement when $|S|=2$,

and then the 2st statement when $|S|=3$,

and so on…

Proof: Base Cases

Base Case 1: 1^{st} statement, $|S|=0$

In this case, we have

$$\Pr(E_k \mid \bigcap_{j \in S} \neg E_j) = \Pr(E_k) \leq p \leq 2p$$

➔ So this case is true

Base Case 2: 2^{nd} statement, $|S|=1$

In this case, we have

$$\Pr(\bigcap_{j \in S} \neg E_j) = 1 - \Pr(E_{s_1}) \geq 1 - p > 0$$

➔ So this case is true

Proof: Inductive Case 1

Inductive Case 1: Assume 1st statement is true for $|S| = 0,1,2,\ldots,h$, and 2nd statement is true for $|S|=1,2,\ldots,h+1$

Then, consider the case when $|S| = h+1$

For a particular E_k, let

$S_1 = \{ j \in S \mid (k,j)$ is an edge in the dependency graph G $\}$

$S_2 = S - S_1$ … [corresponds to mutually independent events]

Note: Since maxdeg(G) $\leq d$, so $|S_1| \leq d$

Proof: Inductive Case 1 (2)

If $|S_2| = |S|$, then E_k is mutually independent of the events $\neg E_j$ for all j in S

In this case:

$$\Pr(E_k \mid \bigcap_{j \in S} \neg E_j) = \Pr(E_k) \leq p \leq 2p$$

Otherwise, $|S_2| < |S|$.

In this case, we introduce a notation:

Let $F_S = \bigcap_{j \in S} \neg E_j$.

Similarly, we define F_{S_1} and F_{S_2}

Proof: Inductive Case 1 (3)

Note: $F_S = F_{S_1} \cap F_{S_2}$

So, $\Pr(E_k \mid \bigcap_{j \in S} \neg E_j)$

$= \Pr(E_k \mid F_S) = \Pr(E_k \cap F_S) / \Pr(F_S)$

$= \Pr(E_k \cap F_{S_1} \cap F_{S_2}) / \Pr(F_{S_1} \cap F_{S_2})$

$= \Pr(E_k \cap F_{S_1} \mid F_{S_2}) \Pr(F_{S_2}) /$

$\Pr(F_{S_1} \mid F_{S_2}) \Pr(F_{S_2})$

$= \Pr(E_k \cap F_{S_1} \mid F_{S_2}) / \Pr(F_{S_1} \mid F_{S_2})$

Proof: Inductive Case 1 (4)

From the previous equality, we have

$$\Pr(E_k \mid \bigcap_{j \in S} \neg E_j)$$

$$= \Pr(E_k \cap F_{S_1} \mid F_{S_2}) / \Pr(F_{S_1} \mid F_{S_2})$$

$$\leq \Pr(E_k \mid F_{S_2}) / \Pr(F_{S_1} \mid F_{S_2})$$

$$= \Pr(E_k) / \Pr(F_{S_1} \mid F_{S_2})$$

$$\leq p / \Pr(F_{S_1} \mid F_{S_2}) \qquad \ldots \text{ (Equation 1)}$$

Proof: Inductive Case 1 (5)

On the other hand, we have

$$\Pr(F_{S_1} \mid F_{S_2}) = \Pr(\bigcap_{j \in S_1} \neg E_j \mid \bigcap_{j \in S_2} \neg E_j)$$

$$= 1 - \Pr(\bigcup_{j \in S_1} E_j \mid \bigcap_{j \in S_2} \neg E_j)$$

$$\geq 1 - \sum_{j \in S_1} \Pr(E_j \mid \bigcap_{j \in S_2} \neg E_j)$$

$\geq 1 - \sum_{j \in S_1} 2p$ ... [by induction hypothesis]

$\geq 1 - 2pd$ ... [since $|S_1| \leq d$]

$\geq 1/2$ ... [since $4pd \leq 1$]

Proof: Inductive Case 1 (6)

So, combining this with Equation 1, we have

$\Pr(E_k \mid \bigcap_{j \in S} \neg E_j)$

$\leq p \,/\, \Pr(F_{S_1} \mid F_{S_2}) \leq 2p$

Thus, 1st statement is true for $|S|=h+1$

➔ This proves Inductive Case 1

It remains to show Inductive Case 2 is true

Proof: Inductive Case 2

Inductive Case 2: Assume 1st and 2nd statement are true for $|S|$ up to $h+1$

Then, consider the case when $|S| = h+2$

$$\Pr(\bigcap_{j \in S} \neg E_j) = \Pr(\bigcap_{j \in \{s_1, s_2, \ldots, s_{h+2}\}} \neg E_j)$$

$$= \prod_{r=1 \text{ to } h+2} \Pr(\neg E_{s_r} \mid \bigcap_{t=1 \text{ to } r-1} \neg E_{s_t})$$

$$= \prod_{r=1 \text{ to } h+2} (1 - \Pr(E_{s_r} \mid \bigcap_{t=1 \text{ to } r-1} \neg E_{s_t}))$$

$$\geq \prod_{r=1 \text{ to } h+2} (1 - 2p) > 0$$ ... [by induction hypothesis]

Conclusion

Thus, 2nd statement is true for $|S|=h+2$

➔ This proves Inductive Case 2

- By induction, we can then show that 2nd statement is true for $|S|=n$
- That is, $\Pr(\bigcap_{j \in S} \neg E_j) > 0$ when $|S|=n$

Consequently, we have

$$\Pr(\text{no BAD events}) = \Pr(\bigcap_{j \in S} \neg E_j) > 0$$

Example: Edge-Disjoint Paths

- There are 50 pairs of users in a network system, each pair wants to obtain a dedicated path for communication
  - That is, they do not want their path to share any edge with the path chosen by others
- Now, we know that each pair has a set of 2000 possible paths to choose, and each such path "crashes" with at most 5 paths in the set of any other pair

Question: Can they get a dedicated path?
Ans. Yes

Edge-Disjoint Paths

In fact, we can show the following based on the Lovasz Local Lemma:

Let F_j = set of m paths pair-j can choose

Theorem: If for all $i \neq j$, each path in F_i "clashes" with no more than k paths in F_j, then, when $8nk/m \leq 1$, there exists a way to choose n edge-disjoint paths connecting the n pairs.

How to prove?

Proof

- Let $E_{i,j}$ = event that paths selected by pair-i and pair-j clashes

  ➔ $\Pr(E_{i,j}) \leq k/m$
- Let G = dependency graph of these events
- Since $E_{i,j}$ is dependent only on events $E_{i,x}$ or $E_{y,j}$ ➔ at most $2n$ events
- Now, by setting $p = k/m$ and $d = 2n$,

$\Pr(E_{i,j}) \leq p$, $\mathrm{maxdeg}(G) \leq d$, and $4pd \leq 1$

➔ We can apply LLL, and theorem follows

Lovasz Local Lemma
(General Case)

Next, we describe the general case of LLL

(the proof is extremely similar to the symmetric case):

Theorem: Let G be a dependency graph of a set of BAD events $\{ E_1, E_2, \ldots, E_n \}$.

Assume that there are $x_1, x_2, \ldots, x_n \in [0,1)$ such that $\Pr(E_i) \leq x_i \prod_{(i,j) \text{ in } G} (1-x_j)$, then

$$\Pr(\text{no BAD events}) \geq \prod_{j=1 \text{ to } n} (1-x_j)$$

Proof

Let $S = \{s_1, s_2, \ldots\}$ be a subset of $\{1, 2, \ldots, n\}$

- The proof is based on induction, where we show two statements are true alternately:

(1) $\Pr(E_k \mid \bigcap_{j \in S} \neg E_j) \leq x_k$ for all S,

with $|S| = 0, 1, 2, \ldots, n-1$

(2) $\Pr(\bigcap_{j \in S} \neg E_j) \geq \prod_{j \in S} (1 - x_j) > 0$

for all S, with $|S| = 1, 2, \ldots, n$

Proof: Base Cases

Base Case 1: 1st statement, $|S|=0$

In this case, we have

$$\Pr(E_k \mid \bigcap_{j \in S} \neg E_j) = \Pr(E_k) \leq x_k \text{ ... [why??]}$$

➔ So this case is true

Base Case 2: 2nd statement, $|S|=1$

In this case, we have

$$\Pr(\bigcap_{j \in S} \neg E_j) = 1 - \Pr(E_{s_1}) \geq 1 - x_{s_1} > 0$$

➔ So this case is true

Proof: Inductive Case 1

Inductive Case 1: Assume 1st statement is true for $|S| = 0,1,2,\ldots,h$, and 2nd statement is true for $|S|=1,2,\ldots,h+1$

Then, consider the case when $|S| = h+1$

For a particular E_k, let

$S_1 = \{ j \in S \mid (k,j)$ is an edge in the dependency graph $G \}$

$S_2 = S - S_1$ … [corresponds to mutually independent events]

Proof: Inductive Case 1 (2)

If $|S_2| = |S|$, then E_k is mutually independent of the events $\neg E_j$ for all j in S

In this case:

$$\Pr(E_k \mid \bigcap_{j \in S} \neg E_j) = \Pr(E_k) \le x_k$$

Otherwise, $|S_2| < |S|$.

In this case:

Let $F_S = \bigcap_{j \in S} \neg E_j$.

Similarly, we define F_{S_1} and F_{S_2}

Proof: Inductive Case 1 (3)

Note: $F_S = F_{S_1} \cap F_{S_2}$

So, $\Pr(E_k \mid \bigcap_{j \in S} \neg E_j)$

$= \Pr(E_k \mid F_S) = \Pr(E_k \cap F_S) \; / \Pr(F_S)$

$= \Pr(E_k \cap F_{S_1} \cap F_{S_2}) \; / \Pr(F_{S_1} \cap F_{S_2})$

$= \Pr(E_k \cap F_{S_1} \mid F_{S_2}) \Pr(F_{S_2}) \; /$

$\Pr(F_{S_1} \mid F_{S_2}) \Pr(F_{S_2})$

$= \Pr(E_k \cap F_{S_1} \mid F_{S_2}) / \Pr(F_{S_1} \mid F_{S_2})$

Proof: Inductive Case 1 (4)

From the previous equality, we have

$$\Pr(E_k \mid \bigcap_{j \in S} \neg E_j)$$

$$= \Pr(E_k \cap F_{S_1} \mid F_{S_2}) / \Pr(F_{S_1} \mid F_{S_2})$$

$$\leq \Pr(E_k \mid F_{S_2}) / \Pr(F_{S_1} \mid F_{S_2})$$

$$= \Pr(E_k) / \Pr(F_{S_1} \mid F_{S_2})$$

$$\leq x_k \prod_{(k,j) \text{ in } G} (1-x_j) / \Pr(F_{S_1} \mid F_{S_2}) \quad \text{... (Equation 1)}$$

Proof: Inductive Case 1 (5)

Now, we label the element of S_1 by $\{y_1, y_2, \ldots, y_r\}$:

$$\Pr(F_{S_1} \mid F_{S_2}) = \Pr(\bigcap_{j \in S_1} \neg E_j \mid \bigcap_{j \in S_2} \neg E_j)$$

$$= \prod_{t=1 \text{ to } r} \Pr(\neg E_{y_t} \mid \bigcap_{v=1 \text{ to } t-1} \neg E_{y_v} \cap \bigcap_{j \in S_2} \neg E_j) \quad **$$

$$= \prod_{t=1 \text{ to } r} \left(1 - \Pr(E_{y_t} \mid \bigcap_{v=1 \text{ to } t-1} \neg E_{y_v} \cap \bigcap_{j \in S_2} \neg E_j)\right)$$

$$\geq \prod_{t=1 \text{ to } r} (1 - x_{y_t}) \quad \text{... [by induction hypothesis]}$$

$$\geq \prod_{(k,j) \text{ in } G} (1 - x_j) \quad \text{... [why??]}$$

Proof: Inductive Case 1 (6)

So, combining this with Equation 1, we have

$\Pr(E_k \mid \bigcap_{j \in S} \neg E_j)$

$\leq x_k \prod_{(k,j) \text{ in } G} (1-x_j) \,/\, \Pr(F_{S_1} \mid F_{S_2}) \leq x_k$

Thus, 1st statement is true for $|S|=h+1$

➔ This proves Inductive Case 1

It remains to show Inductive Case 2 is true

Proof: Inductive Case 2

Inductive Case 2: Assume 1st and 2nd statement are true for |S| up to h+1

Then, consider the case when |S|= h+2

$$\Pr(\bigcap_{j \in S} \neg E_j) = \Pr(\bigcap_{j \in \{s_1, s_2, \ldots, s_{h+2}\}} \neg E_j)$$

$$= \prod_{r=1 \text{ to } h+2} \Pr(\neg E_{s_r} \mid \bigcap_{t=1 \text{ to } r-1} \neg E_{s_t})$$

$$= \prod_{r=1 \text{ to } h+2} (1 - \Pr(E_{s_r} \mid \bigcap_{t=1 \text{ to } r-1} \neg E_{s_t}))$$

$$\geq \prod_{r=1 \text{ to } h+2} (1 - x_{s_r})$$ … [by induction hypothesis]

Conclusion

Since $\prod_{r=1 \text{ to } h+2} (1 - x_{s_r}) = \prod_{j \in S} (1-x_j) > 0$

Thus, 2nd statement is true for $|S|=h+2$

➔ This proves Inductive Case 2

By induction, we can then show that 2nd statement is true for $|S|=n$

Consequently, we have

$$\Pr(\text{no BAD events}) = \Pr\left(\bigcap_{j \in \{1,2,\cdots,n\}} \neg E_j\right)$$

$$\geq \prod_{j=1 \text{ to } n} (1-x_j) > 0$$

Lovasz Local Lemma
(Symmetric Case -- revisited)

The general case can immediately improve the symmetric case by replacing the condition $4pd \leq 1$ to $ep(d+1) \leq 1$, so that we can apply it in more situations

The proof is by setting all $x_i = 1/(d+1)$

→ Then, we can show that

$\Pr(E_i) \leq p \leq x_i \prod_{(i,j) \text{ in } G} (1-x_j)$ ... [how?]

so that we can apply the General Case

(Left as an Exercise)