CS5314
Randomized Algorithms

Lecture 10: Chernoff Bounds
(Introduction)

Objectives

- Chernoff bounds
  - Based on all the moments for bounding tail distribution (such as $\Pr(X \geq a)$)
  - more powerful than Markov (based on $E[X]$) and Chebyshev (based on $E[X]$ and $E[X^2]$)

- Moment generating function

Moment Generating Function

Definition: The moment generating function (MGF) of a random variable X is defined as

$$M_X(t) = E[e^{tX}]$$

Recall: $E[X^k]$ is called the kth moment of X

The MGF is a special function that "captures" all the moments of X

Moment Generating Function

Theorem:

Let X be a random variable with MGF $M_X(t)$

and assume that it is legal to exchange expectation and differentiation operands in calculation **.

Then, for all $n > 1$,

$$E[X^n] = M_X^{(n)}(0),$$

where $M_X^{(n)}(0)$ = nth derivative of $M_X(t)$ evaluated at $t = 0$

** The above assumption is not generally true.
However, it is true for the random variables we study

Proof

$$M_X^{(1)}(t) = \frac{d}{dt}(E[e^{tX}]) = E[\frac{d}{dt}(e^{tX})]$$

$$= E[Xe^{tX}]$$

$$M_X^{(2)}(t) = \frac{d}{dt}(E[Xe^{tX}]) = E[\frac{d}{dt}(Xe^{tX})]$$

$$= E[X^2e^{tX}]$$

…

$$M_X^{(n)}(t) = E[X^ne^{tX}] \rightarrow M_X^{(n)}(0) = E[X^n]$$

Example

Let $X = \mathrm{Geo}(p)$

Questions:

1. What is $E[e^{tX}]$?
2. What is $M_X^{(1)}(t)$? What is $E[X]$?
3. What is $M_X^{(2)}(t)$? What is $E[X^2]$?
4. What is $\mathrm{Var}[X]$?

Equality of MGF

Theorem:

Let X and Y be two random variables.

If there exists $d > 0$ such that

$$M_X(t) = M_Y(t)$$

for all t in $[-d,d]$, then X and Y have the same distribution

Proof: omitted (out of scope) ...

MGF for Sum of Independent RV

Theorem: If X and Y are two independent random variables, then

$$M_{X+Y}(t) = M_X(t)\, M_Y(t)$$

Proof:

$$
\begin{aligned}
M_{X+Y}(t) &= E[e^{t(X+Y)}] \\
&= E[e^{tX}\, e^{tY}] \\
&= E[e^{tX}]\, E[e^{tY}] \qquad \text{... (why?)} \\
&= M_X(t)\, M_Y(t)
\end{aligned}
$$

Remark

Suppose that we know X and Y are two independent random variables, and we have computed $M_X(t)$ and $M_Y(t)$

One day, we find that $M_X(t)\ M_Y(t)$ is an MGF for some random variable Z

What does that imply?

Chernoff Bounds

Theorem: Let X be a random variable.
Then,

(i) $\Pr(X \geq a) \leq \min_{t>0} E[e^{tX}] / e^{ta}$

(ii) $\Pr(X \leq a) \leq \min_{t<0} E[e^{tX}] / e^{ta}$

How to prove?

Proof

(i) For $t > 0$,

$$\begin{aligned} \Pr(X \geq a) &= \Pr(e^{tX} \geq e^{ta}) && \text{... (why?)} \\ &\leq E[e^{tX}] / e^{ta} && \text{... (why?)} \end{aligned}$$

Thus, $\Pr(X \geq a) \leq \min_{t>0} E[e^{tX}] / e^{ta}$ ...(why?)

(ii) Similar proof