CS5314 Randomized Algorithms

Homework 4

Due: 1:10 pm, December 29, 2009 (before class)

1. Let G be a random graph drawn from the $G_{n,p}$ model.
   (a) (10%) What is the expected number of 5-clique in G?
   (b) (10%) What is the expected number of 5-cycle in G?

2. Suppose we have a set of n vectors, $v_1, v_2, \ldots, v_n$, in R^m. Each vector is of unit-length, i.e., $\|v_i\| = 1$ for all i. In this question, we want to show that, there exists a set of values, $\rho_1, \rho_2, \ldots, \rho_n$, each $\rho_i \in \{-1, +1\}$, such that

$$\|\rho_1 v_1 + \rho_2 v_2 + \cdots + \rho_n v_n\| \leq \sqrt{n}.$$

   Intuitively, if we are allowed to "reflect" each v_i as we wish (i.e., by replacing v_i by $-v_i$), then it is possible that the vector formed by the sum of the n vectors is at most $\sqrt{n}$ long.

   (a) (10%) Let $V = \rho_1 v_1 + \rho_2 v_2 + \cdots + \rho_n v_n$, and recall that

$$\|V\|^2 = V \cdot V = \sum_{i,j} \rho_i \rho_j v_i \cdot v_j.$$

   Suppose that each ρ_i is chosen uniformly at random to be -1 or +1. Show that

$$\mathrm{E}[\|V\|^2] = n.$$

   `Hint:`
   - `Can you show that` $\mathrm{E}[\rho_i \rho_j] = 0$ `when` $i \neq j$?
   - `What is the value of` $\mathrm{E}[\rho_i \rho_i]$?
   - `What is the value of` $v_i \cdot v_i$?

   (b) (5%) Argue that there exists a choice of $\rho_1, \rho_2, \ldots, \rho_n$ such that $\|V\| \leq \sqrt{n}$.
   (c) (5%) Your friend, Peter, is more ambitious, and asks if it is possible to to choose $\rho_1, \rho_2, \ldots, \rho_n$ such that

$$\|V\| < \sqrt{n}$$

   instead of $\|V\| \leq \sqrt{n}$ we have just shown. Give a counter-example why this may not be possible.

3. (30%) Let F be a family of subsets of $N = \{1, 2, ..., n\}$. F is called an antichain if there are no $A, B \in F$ satisfying $A \subset B$.

   `Ex:`
   $\{\{1,2,6\},\{3,4\},\{5\}\}$ `is an antichain.`
   $\{\{1,2,6\},\{3,4\},\{2\}\}$ `is not an antichain.` $(\{2\} \subset \{1,2,6\})$

   Let $\sigma \in S_n$ be a random permutation of the elements of N and consider the random variable

$$X = |\{i : \{\sigma(1), \sigma(2), ...\sigma(i)\} \in F\}|$$

By considering the expectation of X prove that $|F| \leq \binom{n}{\lfloor n/2 \rfloor}$.

Hint:
F can be partitioned into n parts according to the size of subsets, like k_i denotes a size-i set. Therefore, $|F| = k_1 + k_2 + ... + k_n$.

4. (30%) Consider a graph in $G_{n,p}$, with $p = 1/n$. Let X be the number of triangles in the graph, where a triangle is a clique with three edges. Show that

$$Pr(X \geq 1) \leq 1/6$$

and that

$$\lim_{n \to \infty} Pr(X \geq 1) \geq 1/7$$

Hint: Use the conditional expectation inequality.