1. Let $G$ be a random graph drawn from the $G_{n,p}$ model.
   
   (a) (10%) What is the expected number of 5-clique in $G$?

   (b) (10%) What is the expected number of 5-cycle in $G$?

2. Suppose we have a set of $n$ vectors, $v_1, v_2, \ldots, v_n$, in $\mathbb{R}^m$. Each vector is of unit-length, i.e., $\|v_i\| = 1$ for all $i$. In this question, we want to show that, there exists a set of values, $\rho_1, \rho_2, \ldots, \rho_n$, each $\rho_i \in \{-1, +1\}$, such that
\[ \|\rho_1v_1 + \rho_2v_2 + \cdots + \rho_nv_n\| \leq \sqrt{n}. \]

   Intuitively, if we are allowed to “reflect” each $v_i$ as we wish (i.e., by replacing $v_i$ by $-v_i$), then it is possible that the vector formed by the sum of the $n$ vectors is at most $\sqrt{n}$ long.

   (a) (10%) Let $V = \rho_1v_1 + \rho_2v_2 + \cdots + \rho_nv_n$, and recall that
\[ \|V\|^2 = V \cdot V = \sum_{i,j} \rho_i\rho_j v_i \cdot v_j. \]

   Suppose that each $\rho_i$ is chosen uniformly at random to be -1 or +1. Show that
\[ \mathbb{E}[\|V\|^2] = n. \]

   Hint:
   
   • Can you show that $\mathbb{E}[\rho_i\rho_j] = 0$ when $i \neq j$?
   • What is the value of $\mathbb{E}[\rho_i\rho_i]$?
   • What is the value of $v_i \cdot v_i$?

   (b) (5%) Argue that there exists a choice of $\rho_1, \rho_2, \ldots, \rho_n$ such that $\|V\| \leq \sqrt{n}$.

   (c) (5%) Your friend, Peter, is more ambitious, and asks if it is possible to to choose $\rho_1, \rho_2, \ldots, \rho_n$ such that
\[ \|V\| < \sqrt{n} \]

   instead of $\|V\| \leq \sqrt{n}$ we have just shown. Give a counter-example why this may not be possible.

3. (30%) Let $F$ be a family of subsets of $N = \{1, 2, \ldots, n\}$. $F$ is called an antichain if there are no $A, B \in F$ satisfying $A \subset B$.

   Ex:
   
   $\{\{1, 2, 6\}, \{3, 4\}, \{5\}\}$ is an antichain.

   $\{\{1, 2, 6\}, \{3, 4\}, \{2\}\}$ is not an antichain. ($\{2\} \subset \{1, 2, 6\}$)

   Let $\sigma \in S_n$ be a random permutation of the elements of $N$ and consider the random variable
\[ X = |\{i : \{\sigma(1), \sigma(2), \ldots, \sigma(i)\} \in F\}| \]
By considering the expectation of $X$ prove that $|F| \leq \binom{n}{\lfloor n/2 \rfloor}$.

Hint:
$F$ can be partitioned into $n$ parts according to the size of subsets, like $k_i$ denotes a size-$i$ set. Therefore, $|F| = k_1 + k_2 + \ldots + k_n$.

4. (30%) Consider a graph in $G_{n,p}$, with $p = 1/n$. Let $X$ be the number of triangles in the graph, where a triangle is a clique with three edges. Show that

$$Pr(X \geq 1) \leq 1/6$$

and that

$$\lim_{n \to \infty} Pr(X \geq 1) \geq 1/7$$

Hint: Use the conditional expectation inequality.