# CS5314 Randomized Algorithms

Homework 3

Due: 1:10 pm, December 8, 2009 (before class)

1. (20%) Let $X$ be a Poisson random variable with mean $\lambda$.
   (a) What is the most likely value of $X$
      i. when $\lambda$ is an integer?
      ii. when $\lambda$ is not an integer?

      *Hint:* Compare $\Pr(X = k + 1)$ with $\Pr(X = k)$.
   (b) We define the median of $X$ to be the least number $m$ such that $\Pr(X \leq m) \geq 1/2$. What is the median of $X$ when $\lambda = 3.9$?

2. (20%) Let $X$ be a Poisson random variable with mean $\mu$, representing the number of criminals in a city. There are two types of criminals: For the first type, they are not too bad and are reformable. For the second type, they are flagrant.

   Suppose each criminal is independently reformable with probability $p$ (so that flagrant with probability $1 - p$). Let $Y$ and $Z$ be random variables denoting the number of reformable criminals and flagrant criminals (respectively) in the city.

   Show that $Y$ and $Z$ are independent Poisson random variables.

3. (20%) Consider assigning some balls to $n$ bins as follows: In the first round, each ball chooses a bin independently and uniformly at random. After that, if a ball lands at a bin by itself, the ball is *served* immediately, and will be removed from consideration. For the number of bins, it remains unchanged.

   In the subsequent rounds, we repeat the process to assign the remaining balls to the bins. We finish when every ball is served.

   (a) Suppose at the start of some round $b$ balls are still remaining. Let $f(b)$ denote the expected number of balls that will remain after this round. Given an explicit formula for $f(b)$.
   (b) Show that $f(b) \leq b^2/n$.

      *Hint:* You may use *Bernoulli's inequality*:

      $$\forall r \in \mathbb{N} \text{ and } x \geq -1, \qquad (1 + x)^r \geq 1 + rx.$$
   (c) Suppose that every round the number of balls served was exactly the expected number of balls to be served. Show that all the balls would be served in $O(\log \log n)$ rounds.

4. (20%) Suppose that we vary the balls-and-bins process as follows. For convenience let the bins be numbered from 0 to $n - 1$. There are $\log_2 n$ players.

   Each player chooses a starting location $\ell$ uniformly at random from $[0, n-1]$ and then places one ball in each of the bins numbered $\ell \bmod n, \ell + 1 \bmod n, \ldots, \ell + n/\log_2 n - 1 \bmod n$. †

   Show that the maximum load in this case is only $O(\log \log n / \log \log \log n)$ with probability that approaches 1 as $n \to \infty$.

† Assume that $n$ is a multiple of $\log_2 n$.

5. (20%) We consider another way to obtain Chernoff-like bound in the balls-and-bins setting without using the theorem in Page 13 of Lecture 14.

   Consider $n$ balls thrown randomly into $n$ bins. Let $X_i = 1$ if the $i$-th bin is empty and 0 otherwise. Let $X = \Sigma_{i=1}^n X_i$.

   Let $Y_i$ be independent Bernoulli random variable such that $Y_i = 1$ with probability $p = (1 - 1/n)^n$. Let $Y = \Sigma_{i=1}^n Y_i$.

   (a) Show that $\mathrm{E}[X_1 X_2 \cdots X_k] \leq \mathrm{E}[Y_1 Y_2 \cdots Y_k]$ for any $k \geq 1$.

   (b) Show that ${X_1}^{j_1} {X_2}^{j_2} \cdots {X_k}^{j_k} = X_1 X_2 \cdots X_k$ for any $j_1, j_2, \ldots, j_k \in \mathbb{N}$.

   (c) Show that $\mathrm{E}[e^{tX}] \leq \mathrm{E}[e^{tY}]$ for all $t \geq 0$.
   *Hint:* Use the expansion for $e^x$ and compare $\mathrm{E}[e^{tX}]$ to $\mathrm{E}[e^{tY}]$.

   (d) Derive a Chernoff bound for $\Pr(X \geq (1 + \delta)\mathrm{E}[X])$.