CS5314 Randomized Algorithms

Homework 2
Due: 13:10, Nov 10, 2009 (before class)

1. (10%) A fixed point of a permutation $\pi : [1, n] \to [1, n]$ is a value for which $\pi(x) = x$. Find the variance in the number of fixed points of a permutation chosen uniformly at random from all permutations.

   Hint: Let X_i be an indicator such that $X_i = 1$ if $\pi(i) = i$. Then, $\sum_{i=1}^{n} X_i$ is the number of fixed points. You cannot use linearity to find $\text{Var}[\sum_{i=1}^{n} X_i]$, but you can calculate it directly.

2. Recall that the covariance of random variables X and Y is:

$$\text{Cov}[X, Y] = \text{E}\left[(X - \text{E}[X])(Y - \text{E}[Y])\right].$$

   We have seen that if X and Y are independent, then the covariance is 0. Interestingly, if X and Y are not independent, the covariance may still be 0.

   (15%) Construct an example where X and Y are not independent, yet $\text{Cov}[X, Y] = 0$.

3. The weak law of large numbers state that, if $X_1, X_2, X_3, ...$ are independent and identically distributed random variables with finite mean μ and finite standard deviation σ, then for any constant $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \Pr\left(\left|\frac{X_1 + X_2 + X_3 + ... + X_n}{n} - \mu\right| > \varepsilon\right) = 0$$

   (15%) Use Chebyshev's inequality to prove the weak law of large numbers.

4. (20%) Suppose you are given a biased coin that has Pr(head) $= p$. Also, suppose that we know $p \geq a$, for some fixed a. Now, consider flipping the coin n times and let n_H be the number of times a head comes up. Naturally, we would estimate p by the value $\tilde{p} = n_H/n$.

   (a) Show that for any $\epsilon \in (0, 1)$,

$$\Pr\Big(|p - \tilde{p}| > \epsilon p\Big) < \exp\left(\frac{-na\epsilon^2}{2}\right) + \exp\left(\frac{-na\epsilon^2}{3}\right)$$

   (b) Show that for any $\epsilon \in (0, 1)$, if

$$n > \frac{3\ln(2/\delta)}{a\epsilon^2},$$

   then

$$\Pr\Big(|p - \tilde{p}| > \epsilon p\Big) < \delta.$$

5. (20%) Let $X_1, X_2, ..., X_n$ be independent Poisson trials such that $\Pr(X_i) = p_i$. Let $X = \sum_{i=1}^{n} X_i$ and $\mu = \text{E}[X]$. During the class, we have learnt that for any $\delta > 0$,

$$\Pr(X \geq (1 + \delta)\mu) < \left(\frac{e^\delta}{(1 + \delta)^{(1+\delta)}}\right)^\mu$$

In fact, the above inequality holds for the *weighted* sum of Poisson trials. Precisely, let $a_1, ..., a_n$ be real numbers in $[0, 1]$. Let $W = \sum_{i=1}^{n} a_i X_i$ and $\nu = \mathrm{E}[W]$. Then, for any $\delta > 0$,

$$\Pr(W \geq (1+\delta)\nu) < \left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}} \right)^{\nu}$$

(a) Show that the above bound is correct.

(b) Prove a similar bound for the probability $\Pr(W \leq (1-\delta)\nu)$ for any $0 < \delta < 1$.

6. (30%) Consider a collection $X_1, X_2, ..., X_n$ of n independent geometric random variables with parameter 1/2. Let $X = \sum_{i=1}^{n} X_i$ and $0 < \delta < 1$.

(a) By applying Chernoff bound to a sequence of $(1+\delta)(2n)$ fair coin tosses,† show that

$$\Pr(X > (1+\delta)(2n)) < \exp\left(\frac{-n\delta^2}{2(1+\delta)} \right).$$

(b) Derive a Chernoff bound on $\Pr(X > (1+\delta)(2n))$ using the moment generating function for geometric random variables as follows:

(i) Show that for $e^t < 2$,

$$\mathrm{E}\left[e^{tX_i}\right] = \frac{e^t}{2 - e^t}.$$

(ii) Show that for $t \in (0, \ln 2)$,

$$\left| \frac{1}{(2-e^t)e^{t(1+2\delta)}} \right| \quad \text{is minimized when } t = \ln\left(1 + \frac{\delta}{(1+\delta)}\right).$$

(iii) Show that

$$\Pr(X > (1+\delta)(2n)) < \left(\left(1 - \frac{\delta}{1+\delta}\right) \left(1 + \frac{\delta}{1+\delta}\right)^{1+2\delta} \right)^{-n}.$$

(c) It is known that when δ is small, there exists $\varepsilon > 0$ such that

$$1 - \frac{\delta}{1+\delta} > e^{-\varepsilon}, \qquad \left(1 + \frac{\delta}{1+\delta}\right)^{(1+\delta)/\delta} > e^{1-\varepsilon}, \quad \text{and} \quad \frac{(1+2\delta)\delta}{1+\delta} > \delta^2.$$

Show that in this case, the bound in 5(b)-(iii) becomes

$$\Pr(X > (1+\delta)(2n)) < \exp\left(-n(1-\varepsilon)\delta^2 - \varepsilon\right).$$

Conclude that when δ is small enough such that ε is arbitrarily close to 0, the above bound is tighter than the bound obtained in 5(a).

7. (Bonus: 10%) Let S be a set of n numbers. The median-finding algorithm discussed in class finds the median of S with high probability, and its running time is $2n + o(n)$. Can you generalize this algorithm so that it can find the kth smallest item of S for any given value of k?

Prove that your resulting algorithm is correct, and bound its running time. (Better bounds may get better grades.)

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†Here, we just assume $(1+\delta)(2n)$ is an integer.