CS5314 Randomized Algorithms

Assignment 1
Due: 1:10 pm, Oct 13, 2009 (before class)

1. (10%) A biased coin is tossed till a head appears for the first time. Suppose that the probability of a head appearing in a toss is p. What is the probability that the number of required tosses is odd?

2. (10%) Suppose Box 1 contains a white balls and b black balls, and Box 2 contains c white balls and d black balls. One ball of unknown color is transferred from the first box into the second one and then a ball is drawn from the latter. What is the probability that it will be a white ball?

3. (20%) A box contains some number of white and black balls. When two balls are drawn without replacement, the probability that both are white is $\frac{1}{3}$.

   (a) Show that $\left(\frac{a-1}{a+b-1}\right)^2 < \frac{1}{3} < \left(\frac{a}{a+b}\right)^2$.

   (b) Show that $\frac{(\sqrt{3}+1)b}{2} < a < 1 + \frac{(\sqrt{3}+1)b}{2}$.

   (c) Find the smallest number of balls in the box.

   (d) How small can the total number of balls be if black balls are even in number?

4. (10%) Let A be an array of n distinct numbers. For any pair (i, j) such that $i < j$ but $A[i] > A[j]$, we say the pair is an *inversion* in A. If each of the $n!$ permutations of the n numbers has equal chance of occurring as A, what is the expected number of inversions?

5. (10%) 20 couples are invited to a party. They are asked to be seated at a long table with 20 seats each side with husbands sitting at one side and wives the other side. If the seating is done at random, what is the expected number of married couples that are seated face to face?

6. (20%) Let X and Y be independent geometric random variables, where X has parameter p and Y has parameter q.

   (a) What is the probability $\Pr(\min(X, Y) = k)$?

   (b) What is $\mathrm{E}[X \mid X \le Y]$?

   (c) What is the probability $\Pr(X = Y)$?

   (d) What is $\mathrm{E}[\max(X, Y)]$?

7. (20%) You need a new staff assistant, and you have n people to interview. You want to hire the best candidate for this position. When you interview the candidates, you can give each of them a score, with the highest score will be the best and no ties being possible.

   You interview the candidates one by one. Because of your company's hiring policy, after you interview the kth candidate, you either offer the candidate the job immediately, or you will forever lose the chance to hire that candidate.

   We suppose that the candidates are interviewed in a random order, chosen uniformly at random from all $n!$ possible orderings.

Consider the following strategy. First, interview m candidates but reject them all. Then from the $(m+1)$th candidate, hire the first one who is better than *all* of the previous candidates you have interviewed.[1]

(a) Let E be the event that we hire the best assistant, and let E_i be the event that ith candidate is the best and we hire him. Determine $\Pr(E_i)$, and show that

$$\Pr(E) = \frac{m}{n} \sum_{j=m+1}^{n} \frac{1}{j-1}.$$

(b) Bound $\sum_{j=m+1}^{n} \frac{1}{j-1}$ to obtain

$$\frac{m}{n}(\log_e n - \log_e m) \leq \Pr(E) \leq \frac{m}{n}(\log_e(n-1) - \log_e(m-1)).$$

(c) Show that $m(\log_e n - \log_e m)/n$ is maximized when $m = n/e$. Explain why this means $\Pr(E) \geq 1/e$ for this choice of m.

8. (Bonus 10%) There may be several different min-cut sets in a graph. Using the analysis of the randomized min-cut algorithm, argue that there can be at most $n(n-1)/2$ distinct min-cut sets.

9. (Bonus 0%) During the class, we have studied a simple randomized algorithm so that given any graph, we can find its min-cut with probability at least $2/(n(n-1))$. Now, we define an r-cut of a graph G to be a set of edges in G whose removal will break G into r or more connected components. (That is, the normal definition of a cut is equivalent to a 2-cut here.) Describe a randomized algorithm for finding an r-cut with minimum number of edges. Also, analyze the probability that the algorithm succeeds in one iteration.

[1] That is, you will hire the kth candidate if $k > m$ and this candidate is better than all of the $k-1$ candidates you have already interviewed.