CS5314
Randomized Algorithms

Tutorial 4: HW 2 Solution

Reminder

Midterm Quiz next Tuesday !!!

Scope: Lectures 1 - 12

Question 1

Definition: Given a permutation π,

if $\pi(x) = x$ [x is not moved by π to somewhere else]

then x is called a fixed point

Let X = total #fixed points in a permutation of $[1,n]$

Question: What is $\mathrm{Var}[X]$?

Question 1

Let X_i = indicator such that:

X_i = 1 if i is a fixed point

X_i = 0 otherwise

➔ $X = X_1 + X_2 + \ldots + X_n$

So, $E[X] = E[X_1 + X_2 + \ldots + X_n]$

$= 1$ (why?)

Question 1

Also, we have

$$E[X^2] = E[(X_1 + X_2 + \dots + X_n)^2]$$

$$= n\, E[X_1^2] + n(n-1)\, E[X_1X_2] \qquad \text{(why?)}$$

$$= n \times 1/n + n(n-1) \times 1/(n(n-1)) \qquad \text{(why?)}$$

$$= 1 + 1 = 2$$

Thus, $\quad \mathrm{Var}[X] = E[X^2] - (E[X])^2$

$$= 2 - 1^2 = 1$$

Question 2 (weak law of large #s)

- Let $X_1, X_2, \ldots, X_n$ be n independent r.v. with same finite mean μ and finite stddev σ

Prove that: For any $\varepsilon > 0$,

$$\lim_{n \to \infty} \Pr\left(|(\textstyle\sum_i X_i/n) - \mu| > \varepsilon \right) = 0$$

Question 2 (weak law of large #s)

Ans. $\lim_{n\to\infty} \Pr(\,|(\Sigma_i X_i/n) - \mu| > \varepsilon)$

$= \quad \lim_{n\to\infty} \Pr(\,|\Sigma_i X_i - n\mu| > n\varepsilon)$

$\leq \quad \lim_{n\to\infty} \Pr(\,(\Sigma_i X_i - n\mu)^2 > (n\varepsilon)^2\,)$

$\leq \quad E[(\Sigma_i X_i - n\mu)^2] \,/\, (n\varepsilon)^2$ (why??)

$= \quad Var[\Sigma_i X_i] \,/\, (n\varepsilon)^2$ (why??)

$= \quad n\sigma^2 \,/\, (n\varepsilon)^2 \quad = 0$ (why??)

Question 3 (Parameter Estimation)

- Suppose we have a biased coin with

$$\Pr(\text{Head}) = p$$

- We don't know the exact value of p, but our friend told us that $p \geq a$ for some a

- As usual, we can perform n flips, and use

$$q = (\#\text{Heads in } n \text{ flips}) / n$$

as an estimate of p

Question 3 (Parameter Estimation)

- Show that for any ε in [0,1]

$$\Pr(|p - q| > \varepsilon p)$$
$$\leq \exp(-n\alpha\varepsilon^2/2) + \exp(-n\alpha\varepsilon^2/3)$$

- Let X = #Heads in n flips = nq
- Also, $X = \mathrm{Bin}(n,p) \rightarrow E[X] = np$

Question 3 (Parameter Estimation)

Ans. $\Pr(|p - q| > \varepsilon p)$

$= \Pr(|np - nq| > n\varepsilon p)$

$= \Pr(np - nq > n\varepsilon p) + \Pr(nq - np > n\varepsilon p)$

$= \Pr(X < E[X](1-\varepsilon)) + \Pr(X > E[X](1+\varepsilon))$

$\leq \exp(-n\alpha\varepsilon^2/2) + \exp(-n\alpha\varepsilon^2/3)$

$\leq 2\exp(-n\alpha\varepsilon^2/3)$

Question 3 (Parameter Estimation)

- Show that for any δ in [0,1]

  if $\quad n > 3 \ln (2/\delta) / (\alpha\varepsilon^2)$

  then $\quad \Pr(|p - q| > \varepsilon p) < \delta$

Ans.

- Follows immediately from last inequality in previous page

Question 4 (weighted Poisson sum)

- Let $X_1, X_2, \ldots, X_n$ be n independent Poisson trials with

$$\Pr(X_i = 1) = p_i$$

- Let $a_1, a_2, \ldots, a_n$ be real numbers in $[0,1]$
- Define

$$W = a_1X_1 + a_2X_2 + \ldots + a_nX_n$$

and $$\nu = E[W] = a_1p_i + a_2p_2 + \ldots + a_np_n$$

Question 4 (weighted Poisson sum)

- Show that

$$\Pr(W \geq (1+\delta)\nu) \leq \left(e^{\delta}/(1+\delta)^{(1+\delta)}\right)^{\nu}$$

Ans. Firstly, for any $t > 0$

$$\Pr(W \geq (1+\delta)\nu)$$

$$= \Pr(e^{tW} \geq e^{t(1+\delta)\nu})$$

$$\leq E[e^{tW}] \,/\, e^{t(1+\delta)\nu}$$

Question 4 (weighted Poisson sum)

Since $E[e^{tW}]$ is the MGF of W, and W is the sum of independent r.v.

→ we can express $E[e^{tW}]$ as

$$E[e^{tW}] = \prod_i E[e^{ta_i X_i}]$$

$$= \prod_i (p_i e^{ta_i} + (1-p_i))$$

$$= \prod_i (1 + p_i (e^{ta_i} - 1))$$

Question 4 (weighted Poisson sum)

Now, because for any r in [0,1],

$$e^{tr} - 1 \leq r(e^t - 1)$$

We have:

$$\begin{aligned} E[e^{tW}] &= \prod_i (1 + p_i (e^{ta_i} - 1)) \\ &\leq \prod_i (1 + p_i a_i (e^t - 1)) \\ &\leq \prod_i e^{p_i a_i (e^t - 1)} \\ &= e^{\nu(e^t - 1)} \end{aligned}$$

Question 4 (weighted Poisson sum)

In other words, for any $t > 0$,

$$\Pr(W \geq (1+\delta)\nu)$$

$$\leq E[e^{tW}] \,/\, e^{t(1+\delta)\nu} \leq e^{\nu(e^t - 1)} \,/\, e^{t(1+\delta)\nu}$$

Then by setting $t = \ln(1+\delta) > 0$,

$$\Pr(W \geq (1+\delta)\nu) \leq \left(e^{\delta}/(1+\delta)^{(1+\delta)}\right)^{\nu}$$

Question 4 (weighted Poisson sum)

- Next, we show that

$$\Pr(W \leq (1-\delta)\nu) \leq \left(e^{-\delta}/(1-\delta)^{(1-\delta)}\right)^{\nu}$$

Ans. Firstly, for any $t < 0$

$$\Pr(W \geq (1-\delta)\nu) \leq E[e^{tW}] \,/\, e^{t(1-\delta)\nu}$$

$$\leq \; e^{\nu(e^{t}-1)} \,/\, e^{t(1-\delta)\nu}$$

➔ Result follows if we set $t = \ln(1-\delta) < 0$

Question 5
(Sum of n Geo(p) versus Bin(n,p))

Let $X_1, X_2, \ldots, X_n$ be n independent Geo(0.5) r.v.

Let $X = X_1 + X_2 + \ldots + X_n$ be their sum

- Show that for any δ in (0,1)

$$\Pr(X \geq (1+\delta)2n) \leq \exp(-n\delta^2/(2(1+\delta)))$$