Randomized algorithm

Tutorial 2

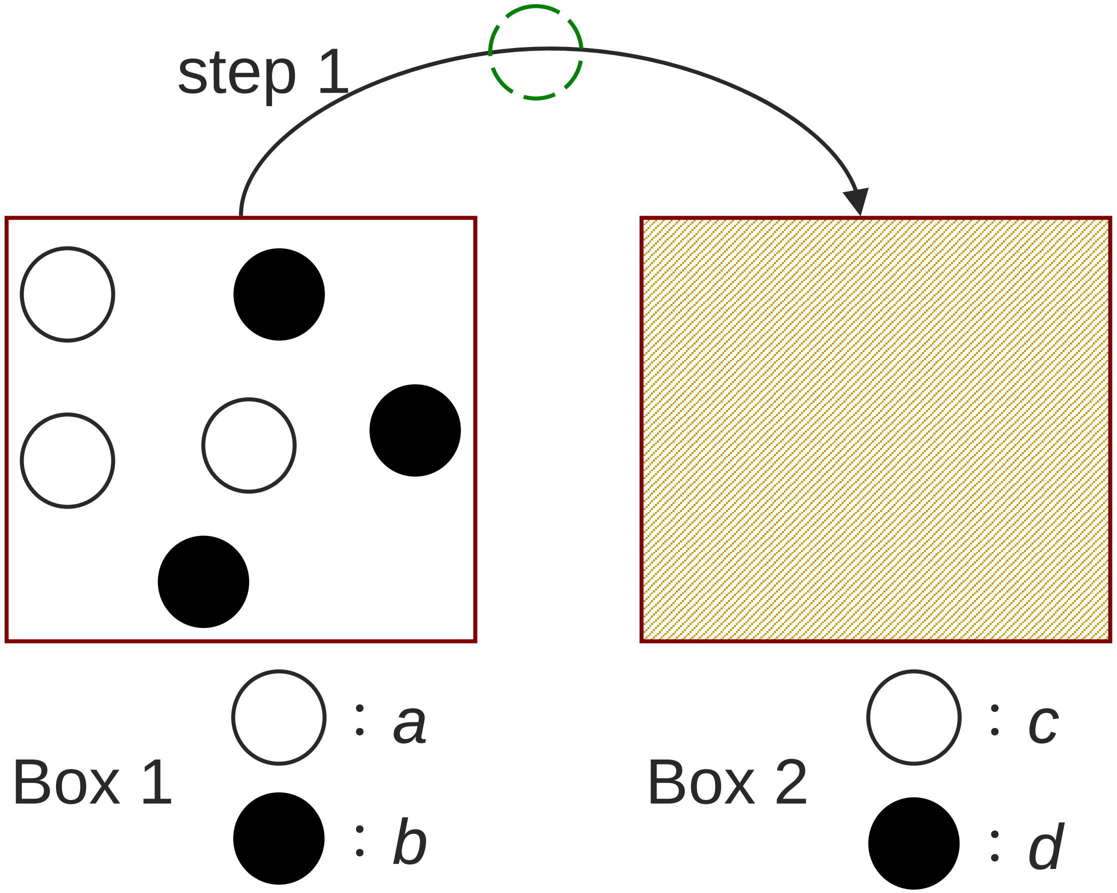
Solution for homework 1

Outline

- Solutions for homework 1
- Negative binomial random variable
- Paintball game
- Rope puzzle

Solutions for homework 1

Homework 1-1



$$\Pr(W) = \frac{a}{a+b}$$

$$\Pr(B) = \frac{b}{a+b}$$

Homework 1-1

- Draw a white ball from Box 2 : event A
  - $\Pr(A)$
    $= \Pr(A \cap (W \cup B))$
    $= \Pr((A \cap W) \cup (A \cap B))$
    $= \Pr(A \cap W) + \Pr(B \cap W)$
    $= \Pr(A|W)\Pr(W) + \Pr(A|B)\Pr(B)$

Homework 1-1

- It is easy to see that

$$\Pr(A \mid W) = \frac{c+1}{c+d+1}$$

$$\Pr(A \mid B) = \frac{c}{c+d+1}$$

$$\Pr(A)$$

$$= \Pr(A \mid W)\Pr(W) + \Pr(A \mid B)\Pr(B)$$

$$= \frac{ac + bc + a}{(a+b)(c+d+1)}$$

Homework 1-2

- W_i = pick a white ball at the *i*-th time

$$\Pr(W_1 \cap W_2)$$
$$= \Pr(W_2 | W_1)\Pr(W_1)$$
$$= \frac{a-1}{a+b-1} \times \frac{a}{a+b}$$
$$= \frac{1}{3}$$

Homework 1-2

$$\frac{a-1}{a+b-1} < \frac{a}{a+b}$$

$$\Rightarrow \left(\frac{a-1}{a+b-1}\right)^2 < \frac{a-1}{a+b-1} \times \frac{a}{a+b} < \left(\frac{a}{a+b}\right)^2$$

$$\Rightarrow \left(\frac{a-1}{a+b-1}\right)^2 < \frac{1}{3} < \left(\frac{a}{a+b}\right)^2$$

Homework 1-2

$$3(a-1)^2 < (a+b-1)^2$$

$$\Rightarrow \sqrt{3}(a-1) < a+b-1$$

$$\Rightarrow (\sqrt{3}-1)a < b-1+\sqrt{3}$$

$$\Rightarrow a < \frac{b+\sqrt{3}-1}{\sqrt{3}-1}$$

$$\Rightarrow a < 1 + \frac{(\sqrt{3}+1)b}{2}$$

Homework 1-2

$$(a+b)^2 < 3a^2$$

$$\Rightarrow a+b < \sqrt{3}a$$

$$\Rightarrow b < (\sqrt{3}-1)a$$

$$\Rightarrow \frac{b}{\sqrt{3}-1} < a$$

$$\Rightarrow \frac{(\sqrt{3}+1)b}{2} < a$$

$$\frac{(\sqrt{3}+1)b}{2} < a < 1 + \frac{(\sqrt{3}+1)b}{2}$$

Homework 1-2

- 1 *b* maps 1 *a* only
- For $b=1$, $1.36<a<2.36$ means $a=2$
  - $Pr(W_1 \cap W_2) = 2/3 * 1/2 = 1/3$

- Then we may try
  - $b=2 \rightarrow a=3 \rightarrow$ odd
  - $b=4 \rightarrow a=6 \rightarrow$ even

Homework 1-3

- By the hint
  - X = sum is even.
  - Y = the number of first die is even
  - Z = the number of second die is odd
  - $\Pr(X \cap Y) = 1/4 = 1/2 * 1/2$
  - $\Pr(X \cap Z) = 1/4 = 1/2 * 1/2$
  - $\Pr(Y \cap Z) = 1/4 = 1/2 * 1/2$
  - $\Pr(X \cap Y \cap Z) = 0$

Homework 1-4

- Let C_1, C_2, …, C_k be all possible min-cut sets.

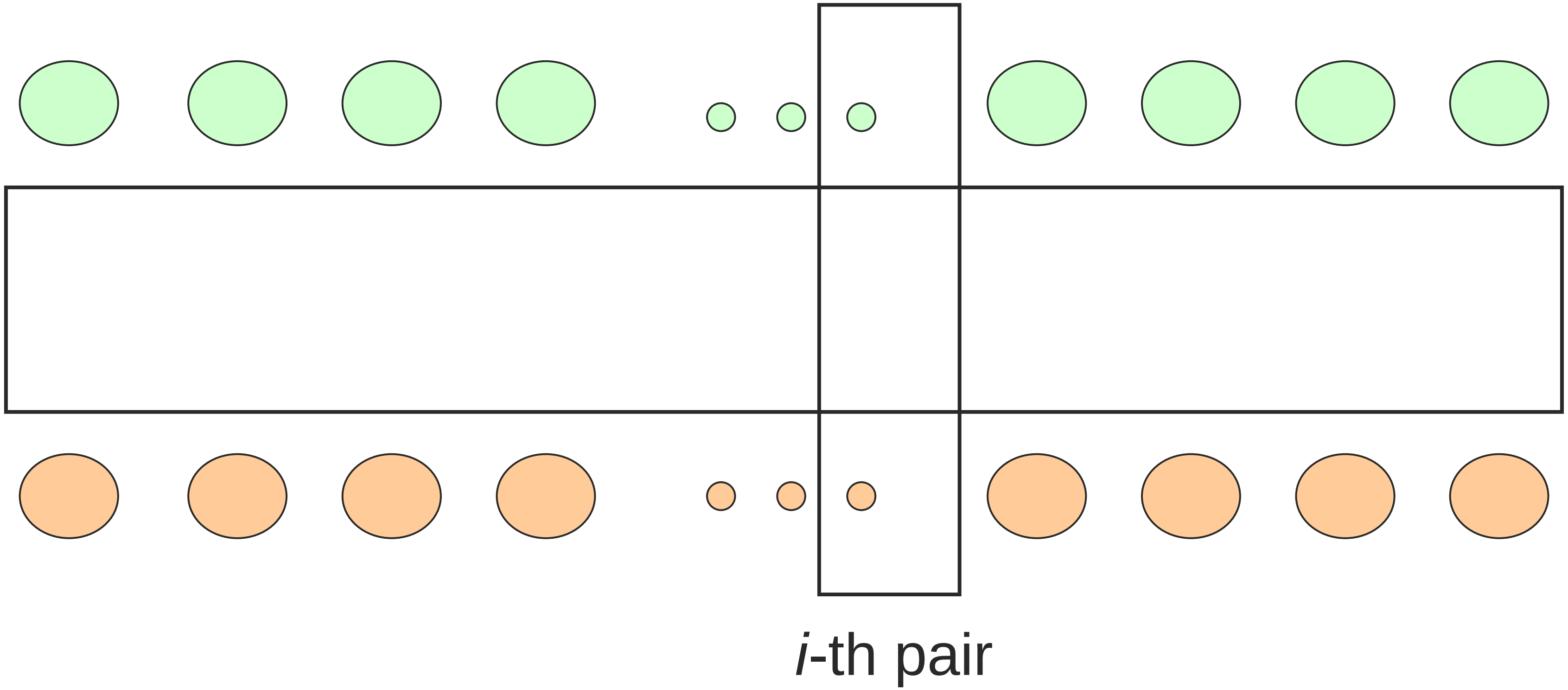
$$\Pr(\text{algorithm returns } C_i) = \frac{2}{n(n-1)}$$

$$\Rightarrow \sum_{i=1}^{k} \Pr(\text{algorithm returns } C_i) \le 1$$

$$\Rightarrow \frac{2k}{n(n-1)} \le 1$$

$$\Rightarrow k \le \frac{n(n-1)}{2}$$

Homework 1-5



Homework 1-5

- Indicator variable
  - $\Pr(X_i) = 1$ if the i-th pair are couple.
  - $\Pr(X_i) = 0$ Otherwise
- Linearity of expectation
  - $X = \sum_{i=1 \text{ to } 20} X_i$
  - $E[X] = \sum_{i=1 \text{ to } 20} E[X_i]$
    $= \sum_{i=1 \text{ to } 20} 1/20$
    $= 1$

Homework 1-6

$$\Pr(X = Y)$$

$$= \sum_k \Pr(X = k \mid Y = k) \Pr(Y = k)$$

$$= \sum_k (1-p)^{1-k} p (1-q)^{1-k} q$$

$$= pq \sum_k (1-p-q+pq)^{1-k}$$

$$= \frac{pq}{p+q-pq}$$

Homework 1-6

$$\mathrm{E}[X] = \frac{1}{p}$$

$$\mathrm{E}[Y] = \frac{1}{q}$$

$$\mathrm{E}[\min(X,Y)] = \frac{1}{1-(1-p)(1-q)}$$

$$\mathrm{E}[\max(X,Y)] = \mathrm{E}[X] + \mathrm{E}[Y] - \mathrm{E}[\min(X,Y)]$$

$$= \frac{1}{p} + \frac{1}{q} - \frac{1}{1-(1-p)(1-q)}$$

Homework 1-7

- To choose the i-th candidate, we need
  - $i>m$
  - i-th candidate is the best [Event B]
  - The best of the first i-1 candidates should be in 1 to m. [Event Y]

Homework 1-7

- Therefore, the probability that *i* is the best candidate is

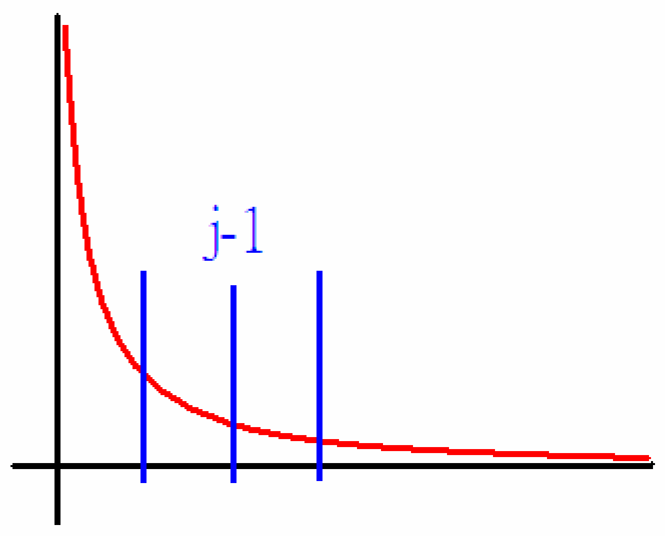
$$\Pr(E_i) = \Pr(B_i)\Pr(Y_i \mid B_i) = \frac{1}{n} \times \frac{m}{i-1}$$

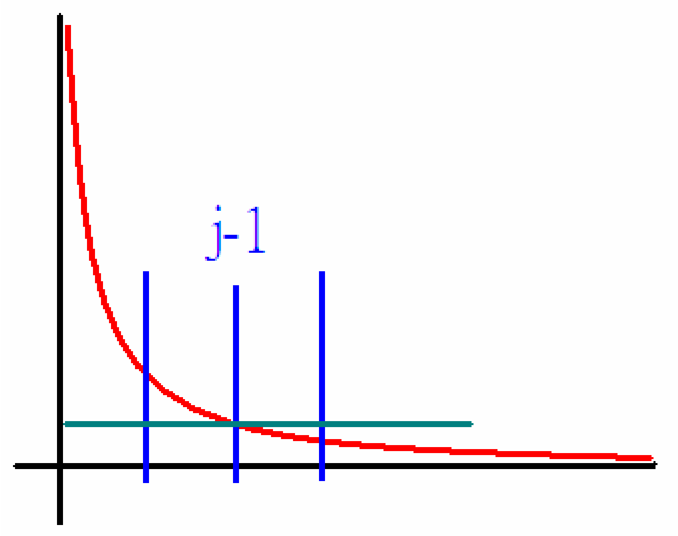
- The probability of selecting the best one is

$$\Pr(E) = \sum_{i=m+1}^{n} \Pr(E_i) = \frac{m}{n} \sum_{i=m+1}^{n} \frac{1}{i-1}$$

Homework 1-7

- Consider the curve f(*x*)=1/*x*. The area under the curve from *x* = *j*-1 to *x* = *j* is less than 1/(*j*-1), but the area from *x* = *j*-2 to *x* = *j*-1 is larger than 1/(*j*-1).





Homework 1-7

$$\sum_{j=m+1}^{n} \frac{1}{j-1} \geq \int_{m}^{n} f(x)dx = \log_e n - \log_e m$$

$$\sum_{j=m+1}^{n} \frac{1}{j-1} \leq \int_{m-1}^{n-1} f(x)dx = \log_e(n-1) - \log_e(m-1)$$

Homework 1-7

$$g(m) = \frac{m}{n}(\log_e n - \log_e m)$$

$$g'(m) = \frac{\log_e n - \log_e m}{n} - \frac{1}{n}$$ → $g'(m)=0$ when $m=n/e$

$$g''(m) = \frac{-1}{mn}$$

→ $g''(m)<0$, $g(m)$ is max when $m=n/e$

Homework 1-7

$$
\begin{aligned}
&\Pr(E) \\
&\geq \frac{m(\log_e n - \log_e m)}{n} \\
&= \frac{n(\log_e n - \log_e (n/e))}{ne} \\
&= \frac{n(\log_e e)}{ne} \\
&= \frac{1}{e}
\end{aligned}
$$

Homework 1-8 (bonus)

- $S=\{\pm1, \pm2, \pm3, \pm4, \pm5, \pm6\}$
- $X=\{1,-2,3,-4,5,-6\}$
- $Y=\{-1,2,-3,4,-5,6\}$
- $E[X]=-0.5$
- $E[Y]=0.5$
- $E[Z] = -3.5$

Negative binomial random variable

Negative binomial random variable

- Definition
  - *y* is said to be negative binomial random variable with parameter *r* and *p* if

$$\Pr(y = k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}$$

  - For example, we want *r* heads when flipping a coin.

Negative binomial random variable

- Let $z=y-r$

$$\Pr(z=k)$$

$$=\Pr(y=k+r)$$

$$=\binom{r+k-1}{r-1}p^r(1-p)^k$$

$$=\binom{r+k-1}{k}p^r(1-p)^k$$

Negative binomial random variable

- Let r=1

$$\Pr(z = k)$$

$$= \binom{r+k-1}{k} p^r (1-p)^k$$

$$= p(1-p)^k$$

- Isn't it familiar?

Negative binomial random variable

- Suppose the rule of meichu game has been changed. The one who win three games first would be the champion. Let *p* be the probability for NTHU to win each game, $0<p<1$.

Negative binomial random variable

- Let the event

$$A_k = \{\text{NTHU wins on the } k\text{-th game}\} \qquad k = 3,4,5$$

- We know

$$\Pr(\text{NTHU wins}) = \Pr\left(\bigcup_{k=3}^{5} A_k\right) = \sum_{k=3}^{5} P(A_k)$$

Negative binomial random variable

- where

$$\Pr(A_k) = P(\text{3rd success on } k\text{th game}) = \binom{k-1}{2} p^3 (1-p)^{k-3}$$

- Hence

$$\Pr(\text{NTHU wins}) = \sum_{k=3}^{5} \binom{k-1}{2} p^3 (1-p)^{k-3}$$

Negative binomial random variable

- Given that NTHU has already won the first game. What is the probability of NTHU being the champion?

$$\sum_{k=2}^{4} \binom{k-1}{1} \left(\frac{1}{2}\right)^{2} \left(\frac{1}{2}\right)^{k-2} = \left(\frac{1}{4} + \frac{2}{8} + \frac{3}{16}\right) = \frac{11}{16}$$

Paintball game

Paintball game

- You James Bond, and Robin Hood decide to play a paintball game. The rules are as follows
  - Everyone attack another in an order.
  - Anyone who get “hit” should quit.
  - The survival is the winner.
  - You can choose shoot into air.

Paintball game

- James Bond is good at using gun, he can hit with probability 100%.
- Robin Hood is a bowman, he can hit with probability 60%.
- You are an ordinary person, can only hit with probability 30%.
- The order of shot is you, Robin Hood then James Bonds.

Paintball game

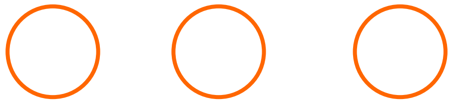
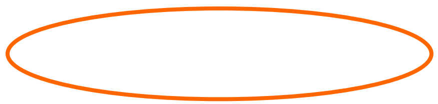
- What is your best strategy of first shoot?
- Please calculate each player’s probability of winning.

Rope puzzle

Rope puzzle

- We have three ropes with equal length (Obviously, there are 6 endpoint).
- Now we randomly choose 2 endpoints and tied them together. Repeat it until every endpoint are tied.

Rope puzzle

- There can be three cases
  1. 
  2. 
  3. 
- Which case has higher probability?