Randomized algorithm

Tutorial 5

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Solution for assignment 4

[Question 1]:

Leader election problem:

We have n users, each with an identifier and want to choose one to be the leader fairly.

Suppose that we have a hash function which outputs a b-bit hash value for each identifier.

Each user obtains the hash value from its identifier, and the leader is the user with the smallest hash value.

Give a lower bound on the number of bits b necessary to ensure that a unique leader is successfully chosen with probability p.

[Solution]:
A unique leader: The smallest bit i which is mapped exactly one user.
This implies that:

$$\begin{aligned} p &= \sum_{i=0}^{2^b-2} \Pr(\text{“bit } i \text{ is mapped exactly by one user”} \\ &\quad \cap \text{“all other users are larger than } i\text{”}) \end{aligned}$$

[Solution]:

$$
\begin{aligned}
p &= \sum_{i=0}^{2^b-2} \frac{1}{2^b} \cdot \left(1 - \frac{i+1}{2^b}\right)^{n-1} \\
&\le \sum_{i=0}^{2^b-2} \frac{1}{2^b} \cdot \left(1 - \frac{1}{2^b}\right)^{n-1} \\
&\le \left(1 - \frac{1}{2^b}\right)^{n-1}
\end{aligned}
$$

By re-arranging terms, we have:

$$
b \ge \log_2 \left(\frac{1}{1 - p^{\frac{1}{n-1}}} \right)
$$

[Question 2]: Let G be a random graph drawn from the $G_{n,1/2}$ model.

1. What is the expected number of 5-clique in G?
2. What is the expected number of 5-cycle in G?

[Solution]:
(a)
Number of 5-vertex set: $\binom{n}{5}$
Probability of a 5-clique : $1/2^{10}$
By the linearity of expectation the expected number of 5-clique is

$$\mathrm{E}[\text{5-clique in } G] = \binom{n}{5} \cdot \frac{1}{2^{10}}$$

[Solution]:
(b)
Number of set: $\binom{n}{5}$
Each 5-vertex set can compose 5! cycles.
Somehow each cycle can be represented in 10 different ways.
Ex: {1,2,3,4,5} {2,3,4,5,1} {3,4,5,1,2} {4,5,1,2,3} {5,1,2,3,4}

So the total number of 5-cycle is

$$\binom{n}{5} \cdot \frac{5!}{10} = 12\binom{n}{5}$$

The probability that a set of 5 vertices a 5-cycle is $1/2^5$, so the expected number of 5-cycle is

$$\mathrm{E}[\text{number of 5-cycle in } G] = 12\binom{n}{5} \cdot \frac{1}{32} = \frac{3}{8} \cdot \binom{n}{5}$$

[Question 3]:Suppose we have a set of n vectors, $v_1, v_2, \ldots, v_n$, in R^m. Each vector is of unit-length, i.e., $\|v_i\| = 1$ for all i. In this question, we want to show that, there exists a set of values, $\rho_1, \rho_2, \ldots, \rho_n$, each $\rho_i \in \{-1, +1\}$, such that

$$\|\rho_1 v_1 + \rho_2 v_2 + \cdots + \rho_n v_n\| \leq \sqrt{n}.$$

Intuitively, if we are allowed to "reflect" each v_i as we wish (i.e., by replacing v_i by $-v_i$), then it is possible that the vector formed by the sum of the n vectors is at most $\sqrt{n}$ long.

(a) Let $V = \rho_1 v_1 + \rho_2 v_2 + \cdots + \rho_n v_n$, and recall that

$$\|V\|^2 = V \cdot V = \sum_{i,j} \rho_i \rho_j v_i \cdot v_j.$$

Suppose that each ρ_i is chosen uniformly at random to be -1 or $+1$. Show that

$$\mathrm{E}[\|V\|^2] = n.$$

(b) Argue that there exists a choice of $\rho_1, \rho_2, \ldots, \rho_n$ such that $\|V\| \leq \sqrt{n}$.

(c) Your friend, Peter, is more ambitious, and asks if it is possible to to choose $\rho_1, \rho_2, \ldots, \rho_n$ such that

$$\|V\| < \sqrt{n}$$

instead of $\|V\| \leq \sqrt{n}$ we have just shown.
Give a counter-example why this may not be possible.

[Solution]:
(a)
Given that

$$\|V\|^2 = V \cdot V,$$

we have

$$\mathrm{E}[\|V\|^2] = \mathrm{E}[V \cdot V] = \mathrm{E}\left[\sum_{i,j} \rho_i \rho_j v_i \cdot v_j\right].$$

When $i \neq j$, ρ_i and ρ_j are independent random variables, so that

$$\mathrm{E}[\rho_i \rho_j] = \mathrm{E}[\rho_i]\mathrm{E}[\rho_j] = 0.$$

On the other hand, when $i = j$, we have

$$\mathrm{E}[\rho_i \rho_i] = 0.5(1)^2 + 0.5(-1)^2 = 1.$$

Also, $v_i \cdot v_i = v_i^2 = 1$ since v_i is a unit vector. Combining the above, we obtain

$$\begin{aligned}
\mathrm{E}[\|V\|^2] &= \mathrm{E}\left[\sum_{i,j} \rho_i \rho_j v_i \cdot v_j\right] \\
&= \mathrm{E}\left[\sum_{i} \rho_i \rho_i v_i \cdot v_i\right] + \mathrm{E}\left[\sum_{i \neq j} \rho_i \rho_j v_i \cdot v_j\right] \\
&= \sum_{i} \mathrm{E}\left[\rho_i \rho_i\right] + \sum_{i \neq j} \mathrm{E}\left[\rho_i \rho_j v_i \cdot v_j\right] \\
&= n + 0 = n.
\end{aligned}$$

(b)
Since $\mathrm{E}[\|V\|^2] = n$, there must be a choice of reflection ρ's such that $\|V\|^2$ is at most n. With that particular choice, $\|V\| \leq \sqrt{n}$.

(c)
Set $v_1 = (1, 0)$ and $v_2 = (0, 1)$. No matter what ρ_1 and ρ_2 is, $\rho_1 \rho_2 v_1 \cdot v_2 = \sqrt{2}$. In general in $\mathbb{R}^m$, we can set

$$v_k = (\underbrace{0, \ldots, 0}_{k-1 \text{ zeroes}}, 1, 0, \ldots, 0)$$

for $k = 1$ to m. Then under any choice of the reflections, the resulting vector V will have length exactly $\sqrt{m}$.

[Question 4]: Consider an instance of SAT with m clauses, where each clause has exactly k literals.
Give a deterministic polynomial-time algorithm that finds an assignment satisfying at least $m(1-2^{-k})$ clauses and analyze its running time.

[Solution]:
Number of satisfied clauses: N_c.
Now we assign values to variables deterministically, one at a time, in an arbitrary order $x_1, x_2, \ldots, x_n$.
Suppose that we have assigned the first k variable. Let $y_1, y_2, \ldots, y_k$ be the corresponding assigned values.

We compute the two quantities,

$$\text{(i)} \qquad E[N_c \mid x_1 = y_1, x_2 = y_2, \ldots, x_k = y_k, x_{k+1} = \mathtt{T}]$$

$$\text{(ii)} \qquad E[N_c \mid x_1 = y_1, x_2 = y_2, \ldots, x_k = y_k, x_{k+1} = \mathtt{F}],$$

and then choose the setting with larger expectation.
As there are n rounds and in each round we need to scan the clauses for only constant number of times, the running time of this algorithm is $O(nmk)$.

[Question 5]:
Use the Lovasz local lemma to show that if

$$4\binom{k}{2}\binom{n}{k-2}2^{1-\binom{k}{2}} \leq 1,$$

then it is possible to color the edges of K_n with two colors so that it has no monochromatic K_k subgraph.

[Solution]:

Consider a random 2-coloring of the graph.

Let E_i be the event that the ith k-set is a monochromatic clique.

Then we have

$$\Pr(E_i) = 2 \cdot \left(\frac{1}{2}\right)^{\binom{k}{2}} = 2^{1-\binom{k}{2}}.$$

Two k-cliques are independent the two cliques share at most one vertex.

For any k clique, there are exactly $\binom{k}{2}\binom{n}{k-2}$ other cliques sharing at least two vertices with it.

Thus, if we construct the dependency graph D for all E_i's, the degree of any node in D can be bounded by

$$d = \binom{k}{2}\binom{n}{k-2}.$$

Thus, whenever

$$4\binom{k}{2}\binom{n}{k-2}2^{1-\binom{k}{2}} \leq 1,$$

we can apply the general Lovasz local lemma and show that there exists a coloring where all E_i's do not occur.
That is, in that coloring, there is no monochromatic k-clique subgraph.

[Question 6]: Use the general form of the Lovasz local lemma to prove that the symmetric version of the Lovasz local lemma can be improved by replacing the condition $4dp \leq 1$ by the weaker condition $ep(d+1) \leq 1$.

The weaker condition allows us to apply Lovasz local lemma even if the bad events are slightly more dependent (larger d), or the bad event may occur with higher probability (larger p).

[Solution]: Choose $x_i = 1/(d+1)$. Since $ep(d+1) \leq 1$, we have

$$\begin{aligned}
x_i \prod_{(i,j)\in E} (1 - x_j) &\geq (1/(d+1))\,(1 - 1/(d+1))^d \\
&\geq ep(1 - 1/(d+1))^d \\
&= ep(d/(d+1))^d \\
&= \frac{ep}{(1+1/d)^d} \\
&\geq ep(1/e) \\
&= p.
\end{aligned}$$

Since $\Pr(E_i) \leq p$ for any E_i, we have

$$\Pr(E_i) \leq x_i \prod_{(i,j) \in E} (1 - x_j).$$

Now we can apply the general Lovasz local lemma, and obtain the desired conclusion that:

$$\Pr(\text{no bad events occur}) > 0.$$

Solution for assignment 5

[Question 1]:

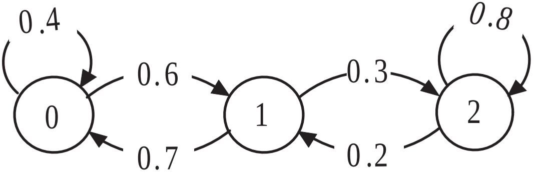

Figure: The modeling Markov chain of this question.

1. Argue that the Markov chain is aperiodic and irreducible.
2. Find the stationary probability.

[Solution]:
(a)Aperiodic:
We define T_i as the number of steps we need to back to state i while we start at state i.

$$T_0 = 1, 2, \cdots$$
$$T_1 = 2, 3, \cdots$$
$$T_2 = 1, 2, \cdots$$

According to the *gcd* of elements in T_0, T_1, and T_2 are 1. This Markov chain is aperiodic.

irreducible:

Now we try to walk from every state and find the possible path.

State 0 $\rightarrow$ State 1.

State 1 $\rightarrow$ State 0 or State 2.

State 2 $\rightarrow$ State 1.

Each state can be connected and form a cycle. This shows that the Markov chain is also irreducible.

(b) Suppose the stationary distribution $p = \{x, y, z\}$. By the model, we may get following equations.

$$x = 0.4x + 0.7y$$

$$y = 0.6x + 0.2z$$

$$z = 0.8z + 0.3y$$

By some calculation, we may get $p = \{7/22, 6/22, 9/22\}$.

[Question 2]:

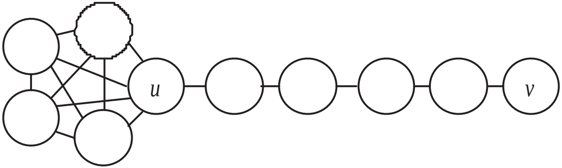

Figure: A lollipop graph.

1. Show that the expected covering time of a random walk starting at v is $\Theta(n^2)$.
2. Show that the expected covering time of a random walk starting at u is $\Theta(n^3)$.

[Solution]:
See the pdf-file.

Thank you