Randomized Algorithm

Tutorial 4

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Solution for Assignment 3
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[Question 1]:

- Let $X \sim \mathrm{Poisson}(\mu)$ denote the number of errors.
- Each error is independently a grammatical error with probability p and a spelling error with probability $1 - p$.
- Let Y and Z be random variables representing the number of grammatical and spelling errors (respectively).

Prove Y and Z are Poisson random variables with means μp and $\mu(1 - p)$, respectively. Also prove that Y and Z are independent.

[Solution]:
We first show that both Y and Z are Poisson random variables.

$$
\begin{aligned}
\Pr(Y=k) &= \sum_{m=k}^{\infty} \Pr(Y=k \mid X=m) Pr(X=m) \\
&= \sum_{m=k}^{\infty} \binom{m}{k} p^k (1-p)^{m-k} \frac{e^{-\mu}\mu^m}{m!} \\
&= \sum_{m=k}^{\infty} \frac{m!}{k!(m-k)!} p^k (1-p)^{m-k} \frac{e^{-\mu}\mu^m}{m!} \\
&= \frac{(\mu p)^k e^{-\mu}}{k!} \sum_{m=k}^{\infty} \frac{(1-p)^{m-k}\mu^{m-k}}{(m-k)!} \\
&= \frac{(\mu p)^k e^{-\mu}}{k!} e^{\mu(1-p)} \quad = \frac{(\mu p)^k e^{-\mu p}}{k!}.
\end{aligned}
$$

By similar steps, we can also get

$$\Pr(Z = k) = (\mu(1-p))^k e^{-(1-p)\mu}/k!.$$

$\Rightarrow Y \sim \mathrm{Poisson}(\mu p)$ and $Z \sim \mathrm{Poisson}(\mu(1-p))$.

Next, we show that Y and X are independent:

$$\begin{aligned}
\Pr(Y = k \cap Z = j) &= \Pr(X = k+j)\Pr(Y = k \mid X = k+j) \\
&= \frac{\mu^{k+j} e^{-\mu}}{(k+j)!}\binom{k+j}{k} p^k (1-p)^j \\
&= \frac{(\mu p)^k e^{-\mu p}}{k!}\frac{(\mu(1-p))^j e^{-\mu(1-p)}}{j!} \\
&= \Pr(Y = k)\Pr(Z = j).
\end{aligned}$$

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&= \Pr(Y = k)\Pr(Z = j).
\end{aligned}$$

[Question 2]:

Let $Z \sim \mathrm{Poisson}(\mu)$, where $\mu \geq 1$ is an integer.

(a) Show that

$$\Pr(Z = \mu + h) \geq \Pr(Z = \mu - h - 1)$$

for $0 \leq h \leq \mu - 1$.

(b) Show that $\Pr(Z \geq \mu) \geq 1/2$.

[Solution]:

(a) By definition, for any non-negative integer k,

$$\Pr(Z = k) = e^{-\mu}\mu^k/k!.$$

Then, we have

$$\begin{aligned}\frac{\Pr(Z = \mu + h)}{\Pr(Z = \mu - h - 1)} &= \frac{\mu^{2h+1}}{(\mu - h)(\mu - (h-1))\cdots(\mu + h)} \\ &= \frac{\mu^2}{\mu^2 - h^2} \cdot \frac{\mu^2}{\mu^2 - (h-1)^2} \cdots \frac{\mu^2}{\mu^2 - 1^2} \cdot \frac{\mu}{\mu} \quad \geq 1.\end{aligned}$$

$\Rightarrow \Pr(Z = \mu + h) \geq \Pr(Z = \mu - h - 1).$

(b) By Part (a), we can easily see that

$$A = \Pr(Z < \mu) = \sum_{k=0}^{\mu-1} \Pr(Z = k) \leq \sum_{k=\mu}^{2\mu-1} \Pr(Z = k) = B.$$

Also, $A + B \leq 1$, so that

$$A \leq 1/2.$$

Thus,

$$\Pr(Z \geq \mu) = 1 - A \geq 1/2.$$

[Question 3]:

Suppose $\mathrm{E}\left[f(X_1^{(m)}, ..., X_n^{(m)})\right]$ is monotonically increasing in m.

(a) Show that

$$\mathrm{E}\left[f(Y_1^{(m)}, ..., Y_n^{(m)})\right] \geq \mathrm{E}\left[f(X_1^{(m)}, ..., X_n^{(m)})\right] \Pr(\sum Y_i^{(m)} \geq m).$$

(b) Then, show that

$$2 \cdot \mathrm{E}\left[f(Y_1^{(m)}, ..., Y_n^{(m)})\right] \geq \mathrm{E}\left[f(X_1^{(m)}, ..., X_n^{(m)})\right].$$

[Solution]:

(a)

$$
\begin{aligned}
& \mathrm{E}\left[f(Y_1^{(m)},\ldots,Y_n^{(m)})\right] \\
= & \sum_{k\geq 0} \mathrm{E}\left[f(Y_1^{(m)},\ldots,Y_n^{(m)}) \mid \sum Y_i^{(m)} = k\right] \Pr(\sum Y_i^{(m)} = k) \\
= & \sum_{k\geq 0} \mathrm{E}\left[f(X_1^{(k)},\ldots,X_n^{(k)})\right] \Pr(\sum Y_i^{(m)} = k) \\
\geq & \sum_{k\geq m} \mathrm{E}\left[f(X_1^{(k)},\ldots,X_n^{(k)})\right] \Pr(\sum Y_i^{(m)} = k) \\
\geq & \sum_{k\geq m} \mathrm{E}\left[f(X_1^{(m)},\ldots,X_n^{(m)})\right] \Pr(\sum Y_i^{(m)} = k) \\
= & \mathrm{E}\left[f(X_1^{(m)},\ldots,X_n^{(m)})\right] \Pr(\sum Y_i^{(m)} \geq m)
\end{aligned}
$$

(b)

- Let $Z = \sum Y_i^{(m)} \Rightarrow Z \sim \mathrm{Poisson}(m)$.
- By Question 1,
$$Pr(Z \geq m) \geq 1/2.$$
- Combining with Part (a), the result follows, since
$$\Pr(\sum Y_i^{(m)} \geq m) = \Pr(Z \geq m) \geq 1/2.$$

[Question 4]:

- Consider n balls thrown randomly into n bins.
- Let $X_i = 1$ if the i-th bin is empty and 0 otherwise.
- Let $X = \sum_{i=1}^{n} X_i$.

We want to get Chernoff bound for the number of empty bins. Instead of using Poisson approximation, we relate X with a binomial random variable Y defined as follows:

- Let Y_i be independent Bernoulli random variable such that $Y_i = 1$ with probability $p = (1 - 1/n)^n$.
- Let $Y = \sum_{i=1}^{n} Y_i$.
- Note: $p = \Pr(X_i = 1)$

(a) Show that for any $k \geq 1$,

$$\mathrm{E}[X_1 X_2 \cdots X_k] \leq \mathrm{E}[Y_1 Y_2 \cdots Y_k].$$

(b) Show that for any $j_1, j_2, \ldots, j_k \in \mathbb{N}$,

$$X_1^{j_1} X_2^{j_2} \cdots X_k^{j_k} = X_1 X_2 \cdots X_k.$$

(c) Show that for all $t \geq 0$,

$$\mathsf{E}[e^{tX}] \leq \mathsf{E}[e^{tY}].$$

(d) Derive a Chernoff bound for $\Pr(X \geq (1+\delta)\mathsf{E}[X])$.

(a)
$$\begin{aligned}\mathrm{E}[X_1 X_2 \cdots X_k] &= \Pr(X_1 = 1 \cap X_2 = 1 \cap \ldots \cap X_k = 1) \\ &= \Pr(\text{each ball is in the other } n-k \text{ bins}) \\ &= \left(\frac{n-k}{n}\right)^n .\end{aligned}$$

$$\mathrm{E}[Y_1 Y_2 \cdots Y_k] = \Pr(Y_1 = 1 \cap Y_2 = 1 \cap \ldots \cap Y_k = 1) = \left(1 - \frac{1}{n}\right)^{kn} .$$

By Bernoulli's inequality, we have

$$\left(\frac{n-k}{n}\right) = 1 - \frac{k}{n} \leq \left(1 - \frac{1}{n}\right)^k .$$

$$\Rightarrow \quad \mathrm{E}[X_1 X_2 \cdots X_k] \leq \mathrm{E}[Y_1 Y_2 \cdots Y_k].$$

(a) An alternative solution (by induction):

$$\begin{aligned}
\mathrm{E}[X_1X_2\cdots X_kX_{k+1}] &= \mathrm{E}[X_1X_2\cdots X_k \mid X_{k+1}=1]\Pr(X_{k+1}=1) \\
&\leq \mathrm{E}[X_1X_2\cdots X_k]\Pr(X_{k+1}=1) \qquad \text{(why?)} \\
&\leq \left(1-\frac{1}{n}\right)^{(k+1)n} \\
&= \mathrm{E}[Y_1Y_2\cdots Y_kY_{k+1}].
\end{aligned}$$

(b) Since X_i is an indicator, we have:

1. If $X_i = 1$ for all i,

$$X_1^{j_1} X_2^{j_2} \cdots X_k^{j_k} = X_1 X_2 \cdots X_k = 1;$$

2. Otherwise, $X_i = 0$ for some i, so that

$$X_1^{j_1} X_2^{j_2} \cdots X_k^{j_k} = X_1 X_2 \cdots X_k = 1;$$

Thus, in any case, $X_1^{j_1} X_2^{j_2} \cdots X_k^{j_k} = X_1 X_2 \cdots X_k$.

(c) $\mathrm{E}[e^{tX}] = \mathrm{E}[1 + tX + \frac{(tX)^2}{2!} + \ldots] = \mathrm{E}[1] + t\mathrm{E}[X] + \frac{t^2}{2!}\mathrm{E}[X^2] + \cdots$

By (b), we can show that for all integer $r \geq 1$,

$$\mathrm{E}[X^r] = \mathrm{E}[(X_1 + X_2 + \cdots + X_n)^r] \leq \mathrm{E}[(Y_1 + Y_2 + \cdots + Y_n)^r] = \mathrm{E}[Y^r]$$

For instance, when $r = 2$,

$$\begin{aligned} \mathrm{E}[X^2] &= \mathrm{E}[(X_1 + X_2 + \cdots + X_n)^2] = \mathrm{E}\left[\sum_i X_i^2 + \sum_{j \neq k} X_j X_k\right] \\ &\leq \mathrm{E}\left[\sum_i Y_i^2 + \sum_{j \neq k} Y_j Y_k\right] = \mathrm{E}[(Y_1 + Y_2 \cdots + Y_n)^2] = \mathrm{E}[Y^2]. \end{aligned}$$

Thus, $\mathrm{E}[e^{tX}] \leq \mathrm{E}[e^{tY}]$

(c) $\mathrm{E}[e^{tX}] = \mathrm{E}[1+tX+\frac{(tX)^2}{2!}+...] = \mathrm{E}[1]+t\mathrm{E}[X]+\frac{t^2}{2!}\mathrm{E}[X^2]+\cdots$

By (b), we can show that for all integer $r \geq 1$,

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For instance, when $r = 2$,

$$\begin{aligned}
\mathrm{E}[X^2] &= \mathrm{E}[(X_1+X_2+\cdots+X_n)^2] = \mathrm{E}\left[\sum_i X_i^2 + \sum_{j\neq k} X_j X_k\right] \\
&\leq \mathrm{E}\left[\sum_i Y_i^2 + \sum_{j\neq k} Y_j Y_k\right] = \mathrm{E}[(Y_1+Y_2\cdots+Y_n)^2] = \mathrm{E}[Y^2].
\end{aligned}$$

Thus, $\mathrm{E}[e^{tX}] \leq \mathrm{E}[e^{tY}]$

(d) Let $\mu = \mathrm{E}[X] = \mathrm{E}[Y] = np$. So for any $t > 0$, we have:

$$\Pr(X \geq (1+\delta)\mathrm{E}[X]) \leq \frac{\mathrm{E}[e^{tX}]}{e^{t(1+\delta)\mu}} \leq \frac{\mathrm{E}[e^{tY}]}{e^{t(1+\delta)\mu}}$$

On the other hand,

$$\mathrm{E}[e^{tY}] = ((1-p) + pe^t)^n \leq e^{np(e^t-1)} \leq e^{(e^t-1)\mu}$$

Combining, and then setting $t = \ln(1+\delta)$, we get

$$Pr(X \geq (1+\delta)\mathrm{E}[X]) \leq \frac{e^{(e^t-1)\mu}}{e^{(1+\delta)t\mu}} = \left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right)^{\mu}$$

Thank you