1. Ans.

   (a) We define $T_i$ as the number of steps we can return to state $i$ while we start at state $i$. Then some of the possible values of each $T_i$ are listed as follows:

   $T_0 = 1, 2, \cdots$

   $T_1 = 2, 3, \cdots$

   $T_2 = 1, 2, \cdots$

   Since the $gcd$ of elements in $T_0$, $T_1$, and $T_2$ are all 1, this implies that the Markov chain is aperiodic.

   On the other hand, we can easily find a cycle traversing each state at least once. Thus, the graph is strongly connected, so that the Markov chain is also irreducible.

   (b) Since the Markov chain is aperiodic and irreducible, a stationary distribution exists which is also unique. Suppose such a stationary distribution is $p = \langle x, y, z \rangle$. By the transition probabilities, we get following equations.

   $x = 0.4x + 0.7y$

   $y = 0.6x + 0.2z$

   $z = 0.8z + 0.3y$

   By some calculation, we get $p = \langle \frac{7}{22}, \frac{6}{22}, \frac{9}{22} \rangle$.

2. Ans. We use the following notation for random walk on any undirected graph $G = (V, E)$:

   - For any two vertices $u, v \in V$, $H_{u,v}$ denotes the expected hitting time from $u$ to $v$.
   - For any vertex $v \in V$, $C_v(G)$ denotes the cover time from $v$.
   - $C(G) = \max_v C_v(G)$.

   (a) Lower bound:

   Since $H_{v,u} = \Theta(n^2)$ (see Lecture 21 pg. 24), we get $C_v(G) = \Omega(n^2)$.

   Upper bound:

   Let $w$ be the neighbor node of $u$ on the path from $u$ to $v$. Let $x$ be a node on the clique which is not $u$.

   $C_v(G) \leq H_{v,u} + \text{starting at } u \text{ and visit all vertices of the clique} \leq H_{v,u} + 1/(n/2) \times (1 + C_{v}(G)) + (n/2 - 1)/(n/2) \times (1 + C_x(G)) \leq H_{v,u} + 1/(n/2) \times (1 + C(G)) + (n/2 - 1)/(n/2) \times (1 + C_x(G))$
expected steps leaving the clique from \( u \) and coming back to \( u \) \( ) \)

\[
H_{v,u} + 1/(n/2) \times O(n^3)
\]

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\[
+(n/2 - 1)/(n/2) \times (\Theta(n \log n))
\]

coupon collecting problem

\[
\Theta(n \log n) \times (1/n) \times 1/(n/2) \times O(n^2)
\]

expected times hitting \( u \)

\[
O(n^2)
\]

expected times leaving the clique

\[
\]

expected steps coming back to \( u \)

\[
= O(n^2).
\]

By lower bound and upper bound, the expected covering time of a random walk starting at \( v \) is \( \Theta(n^2) \)

(b) Upper bound:

The \( C_u(G) \) is bounded above by \( O(|V| \cdot |E|) = O(n^3) \) (see Lecture 23, Page 28).

Lower bound:

Since covering the whole graph must take more time than covering part of the graph, 
\( C_u(G) \geq H_{u,v} \).

Let \( w \) be the neighbor node of \( u \) on the path from \( u \) to \( v \). Let \( x \) be a node on the clique which is not \( u \).

The value \( H_{u,v} \) can be expressed as

\[
H_{u,v} = 1/(n/2)(1 + H_w,v) + (1 - 1/(n/2))(1 + H_x,v).
\]

Let \( p \) be the probability that we reach \( u \) before reaching \( v \), starting from \( w \). Obviously, we have

\[
H_{w,v} \geq pH_{u,v}.
\]

From the Gambler’s Ruin Problem, we can calculate \( p \) to be \( (1 - 2/n) \) (by considering a fair-coin game of two players whose initial capitals are \( n/2 - 1 \) dollars and 1 dollar respectively). Hence,

\[
H_{w,v} \geq (1 - 2/n) H_{u,v}.
\]

Let \( r \) be the expected number of steps to reach \( u \), starting from another vertex \( x \) in the clique. We can calculate \( r = n/2 \) since the number of steps is a geometric random variable with parameter \( 1/(n/2) \).

Thus, the value \( H_{x,v} \) can be expressed by:

\[
H_{x,v} = H_{u,v} + \Omega(n).
\]

Combining everything, we have

\[
H_{u,v} = \frac{1}{n/2}(1 + H_w,v) + (1 - \frac{1}{n/2})(1 + H_x,v)
\]

\[
\geq \frac{1}{n/2} \left( 1 + (1 - \frac{2}{n})H_{u,v} \right) + (1 - \frac{1}{n/2}) (1 + H_{u,v} + \Omega(n))
\]

\[
= \left( 1 - \frac{4}{n^2} \right) H_{u,v} + 1 + \Omega(n)
\]
By re-arranging terms, we get \( H_{u,v} = \Omega(n^3) \), which implies
\[
C_u(G) = \Omega(H_{u,v}) = \Omega(n^3).
\]

By lower bound and upper bound, the expected covering time of a random walk starting at \( u \) is \( \Theta(n^3) \).