CS5314 Randomized Algorithms

Homework 4

Due: 1:10 pm, December 30, 2008 (before class)

1. (20%) For the leader election problem briefly introduced in Lecture Notes 15, we have n users, each with an identifier. Suppose that we have a *good* hash function (that looks uniform and independent), which outputs a b-bit hash value for each identifier. One way to solve the leader election problem is as follows: Each user obtains the hash value from its identifier, and the leader is the user with the smallest hash value.

   Give a lower bound on the number of bits b necessary to ensure that a unique leader is successfully chosen with probability p. Make your bound as tight as possible.

2. Let G be a random graph drawn from the $G_{n,1/2}$ model.

   (a) (10%) What is the expected number of 5-clique in G?

   (b) (10%) What is the expected number of 5-cycle in G?

3. Suppose we have a set of n vectors, $v_1, v_2, \ldots, v_n$, in R^m. Each vector is of unit-length, i.e., $\|v_i\| = 1$ for all i. In this question, we want to show that, there exists a set of values, $\rho_1, \rho_2, \ldots, \rho_n$, each $\rho_i \in \{-1, +1\}$, such that

$$\|\rho_1 v_1 + \rho_2 v_2 + \cdots + \rho_n v_n\| \leq \sqrt{n}.$$

   Intuitively, if we are allowed to "reflect" each v_i as we wish (i.e., by replacing v_i by $-v_i$), then it is possible that the vector formed by the sum of the n vectors is at most $\sqrt{n}$ long.

   (a) (10%) Let $V = \rho_1 v_1 + \rho_2 v_2 + \cdots + \rho_n v_n$, and recall that

$$\|V\|^2 = V \cdot V = \sum_{i,j} \rho_i \rho_j v_i \cdot v_j.$$

   Suppose that each ρ_i is chosen uniformly at random to be -1 or +1. Show that

$$\mathrm{E}[\|V\|^2] = n.$$

   Hint: (i) Can you show that $\mathrm{E}[\rho_i \rho_j] = 0$ when $i \neq j$? (ii) What is the value of $\mathrm{E}[\rho_i \rho_i]$? (iii) What is the value of $v_i \cdot v_i$?

   (b) (5%) Argue that there exists a choice of $\rho_1, \rho_2, \ldots, \rho_n$ such that $\|V\| \leq \sqrt{n}$.

   (c) (5%) Your friend, Peter, is more ambitious, and asks if it is possible to to choose $\rho_1, \rho_2, \ldots, \rho_n$ such that

$$\|V\| < \sqrt{n}$$

   instead of $\|V\| \leq \sqrt{n}$ we have just shown. Give a counter-example why this may not be possible.

4. (20%) Consider an instance of SAT with m clauses, where each clause has exactly k literals. Give a deterministic polynomial-time algorithm that finds an assignment satisfying at least $m(1 - 2^{-k})$ clauses and analyze its running time.

5. (20%) Use the Lovasz local lemma to show that if

$$4\binom{k}{2}\binom{n}{k-2}2^{1-\binom{k}{2}} \leq 1,$$

then it is possible to color the edges of K_n with two colors so that it has no monochromatic K_k subgraph.

Remark. This bound is slightly better than the bound in Page 4 of Lecture Notes 17.[**]

6. (0%) Use the general form of the Lovasz local lemma to prove that the symmetric version of the Lovasz local lemma can be improved by replacing the condition $4dp \leq 1$ by the weaker condition $ep(d+1) \leq 1$.[§]

[**]Recall that as n increases, it is becoming more difficult to find a two-coloring for K_n with no monochromatic K_k. Precisely, when n is large enough, *any* two-coloring of K_n *must* contain a monochromatic K_k. The bound in Q4 is better in a sense that it shows such a two-coloring exists even if n is slightly larger.

[§]Roughly speaking, the weaker condition allows us to apply Lovasz local lemma even if the bad events are slightly more dependent (larger d), or the bad event may occur with higher probability (larger p).