CS5314 Randomized Algorithms

Homework 3

Due: 2:10 pm, November 20, 2008 (before class)

1. (25%) Let X be a Poisson random variable with mean μ, representing the number of errors on a page of this book. Each error is independently a grammatical error with probability p and a spelling error with probability $1-p$. If Y and Z be random variables representing the number of grammatical and spelling errors (respectively) on a page of this book, prove that Y and Z are Poisson random variables with means μp and $\mu(1-p)$, respectively. Also prove that Y and Z are independent.

2. (25%) Let Z be a Poisson random variable of mean μ, where $\mu \geq 1$ is an integer.

   (a) Show that $\Pr(Z = \mu + h) \geq \Pr(Z = \mu - h - 1)$ for $0 \leq h \leq \mu - 1$.

   (b) Using part (a), argue that $\Pr(Z \geq \mu) \geq 1/2$.

   Hint: Use the definition of Poisson distribution $\Pr(Z = j) = e^{-\mu}\mu^j/j!$.

3. (25%) In Page 14 of Lecture Notes 14 we showed that, for any nonnegative functions f,

$$\mathrm{E}[f(Y_1^{(m)}, ..., Y_n^{(m)})] \geq \mathrm{E}[f(X_1^{(m)}, ..., X_n^{(m)})]\Pr(\Sigma Y_i^{(m)} = m)$$

   (a) Now suppose we further know that $\mathrm{E}[f(X_1^{(m)}, ..., X_n^{(m)})]$ is monotonically increasing in m. Show that

$$\mathrm{E}[f(Y_1^{(m)}, ..., Y_n^{(m)})] \geq \mathrm{E}[f(X_1^{(m)}, ..., X_n^{(m)})]Pr(\Sigma Y_i^{(m)} \geq m)$$

   (b) Combining part (a) and part (b) with the results in Question 2, prove the following theorem, which appears in Page 20 of Lecture Notes 14:

   Theorem: Let $f(x_1, ..., x_n)$ be a nonnegative function such that $\mathrm{E}[f(X_1^{(m)}, ..., X_n^{(m)})]$ is monotonically increasing in m. Then

$$\mathrm{E}[f(X_1^{(m)}, ..., X_n^{(m)})] \leq 2\mathrm{E}[f(Y_1^{(m)}, ..., Y_n^{(m)})]$$

4. (25%) We consider another way to obtain Chernoff-like bound in the balls-and-bins setting.

   Consider n balls thrown randomly into n bins. Let $X_i = 1$ if the i-th bin is empty and 0 otherwise. Let $X = \Sigma_{i=1}^n X_i$.

   Let Y_i be independent Bernoulli random variable such that $Y_i = 1$ with probability $p = (1 - 1/n)^n$. Let $Y = \Sigma_{i=1}^n Y_i$.

   (a) Show that $\mathrm{E}[X_1 X_2 \cdots X_k] \leq \mathrm{E}[Y_1 Y_2 \cdots Y_k]$ for any $k \geq 1$.

   (b) Show that ${X_1}^{j_1}{X_2}^{j_2} \cdots {X_k}^{j_k} = X_1 X_2 \cdots X_k$ for any $j_1, j_2, \ldots, j_k \in \mathbb{N}$.

   (c) Show that $\mathrm{E}[e^{tX}] \leq \mathrm{E}[e^{tY}]$ for all $t \geq 0$.

   Hint: Use the expansion for e^x and compare $\mathrm{E}[e^{tX}]$ to $\mathrm{E}[e^{tY}]$.

   (d) Derive a Chernoff bound for $\Pr(X \geq (1 + \delta)\mathrm{E}[X])$.