CS5314 Randomized Algorithms

Homework 2

Due: 1:10 pm, November 11, 2008 (before class)

1. (20%) Let S be a set of n numbers. The median-finding algorithm discussed in class finds the median of S with high probability, and its running time is $2n + o(n)$.

   Can you generalize this algorithm so that it can find the kth largest item of S for any given value of k? Prove that your resulting algorithm is correct, and bound its running time. (Better bounds may get better grades.)

2. (20%) The weak law of large numbers state that, if $X_1, X_2, X_3, ...$ are independent and identically distributed random variables with finite mean μ and finite standard deviation σ, then for any constant $\varepsilon > 0$ we have

$$\lim_{n \to \infty} \Pr\left(\left| \frac{X_1 + X_2 + X_3 + ... + X_n}{n} - \mu \right| > \varepsilon \right) = 0$$

   Use Chebyshev's inequality to prove the weak law of large numbers.

3. (20%)

   (a) Determine the moment generating function for the binomial random variable $Bin(n, p)$.

   (b) Let X be a $Bin(n, p)$ random variable and Y be a $Bin(m, p)$ random variable. Suppose that X and Y are independent. Use part (a) to determine the moment generating function of $X + Y$.

   (c) What can we conclude from the form of the moment generating function of $X + Y$?

4. (20%) In a perfect wireless communication system, a receiver can listen to a dedicated sender, and receive properly any bit sent by the sender. The bit $b(t)$ sent at time t can be represented by a 1 or -1. Unfortunately, in the real situation, there are noise coming from other nearby communications, which can affect the receiver's signal. A simplified model to include the effect of noise is as follows:

   > Let n be the number of other senders. For the ith such sender, it has strength $0 \leq p_i \leq 1$; at any time t, this sender sends a bit $b_i(t)$ independently, where $b_i(t)$ has equal chance to be 1 or -1.
   >
   > The receiver obtains the signal $s(t)$ given by
   >
   > $$s(t) = b(t) + \sum_{i=1}^{n} p_i b_i(t).$$
   >
   > If $s(t)$ is closer to 1 than -1, the receiver assumes that the bit sent at time t was a 1; otherwise, the receiver assumes that it was a -1.

   Give a Chernoff bound to estimate the probability that the receiver makes an error in determining $b(t)$.

5. (20%) For this question, you may directly assume the following theorem is proven:

Theorem 1. *Let $X_1, X_2, \ldots, X_n$ be any mutually independent random variables, with $0 \le X_i \le 1$ for each i. Let $X = \sum_{i=1}^{n} T_i$. Then for any $c > 1$,*

$$\Pr(X \ge c\mathrm{E}[X]) \le e^{-\alpha \mathrm{E}[X]},$$

where $\alpha = c \ln c + 1 - c$.

Let $J_1, J_2, \cdots, J_n$ be a set of jobs, where the i-th job J_i has an execution time of L_i seconds, $0 \le L_i \le 1$. Suppose we have m servers, and we can assign the n jobs to run on them. The *load* of a server is the total execution time of all jobs assigned to it, and our goal is to find an assignment so that the load is as balanced as possible.

Unfortunately, this problem is NP-hard, so that no efficient (polynomial-time) algorithm is known that finds the assignment. However, surprisingly, in this question, we show that by simply assigning jobs independently and uniformly at random to the servers, we can still obtain good performance guarantee.

(a) Let $\bar{L}$ denote the expected load of each server. Express $\bar{L}$ in terms of L_i's.

(b) By using union bound, show that the probability that some server has load more than $c\bar{L}$ is at most $me^{-\alpha\bar{L}}$, where $\alpha = c \ln c + 1 - c$.

(c) Suppose $n = 100K$, $m = 10$, and the average job execution time is 0.25 seconds. Show that

$$\Pr(\text{all servers have load at most } 1.1\bar{L}) \ge 1 - e^{-9}.$$

6. (Bonus: 10%) This question attempts to prove Theorem 1. Part (a) establishes a useful fact, while the remaining parts proceed to give the proof.

(a) Let c and λ be two positive real numbers, with $c > 1$ and $0 \le \lambda \le 1$. Show that for any $z \in [-\lambda, 1-\lambda]$,

$$c^z \le c^{-\lambda}(1 + \lambda(c-1)) + z(c^{1-\lambda} - c^{-\lambda}).$$

Furthermore, argue that

$$c^z \le c^{-\lambda} e^{\lambda(c-1)} + z(c^{1-\lambda} - c^{-\lambda}).$$

Hint. Let

$$f(z) = c^{-\lambda}(1 + \lambda(c-1)) + z(c^{1-\lambda} - c^{-\lambda}) - c^z.$$

Compute $f'(z)$ and conclude that $f(z) \ge 0$ for all $z \in [-\lambda, 1-\lambda]$.

(b) Next, we define $\Delta_i = X_i - \mathrm{E}[X_i]$.[†] Thus, we have

$$-\mathrm{E}[X_i] \le \Delta_i \le 1 - \mathrm{E}[X_i] \quad \text{and} \quad \mathrm{E}[\Delta_i] = \mathrm{E}\left[X_i - \mathrm{E}[X_i]\right] = 0.$$

Let $\Delta = \sum_{i=1}^{n} \Delta_i$. Show that

$$\Pr(X \ge c\mathrm{E}[X]) \;=\; \Pr(\Delta \ge (c-1)\mathrm{E}[X]) \;\le\; \frac{\mathrm{E}[c^{\Delta}]}{c^{(c-1)\mathrm{E}[X]}}.$$

(c) Using Part (a), show that if we set $\lambda = \mathrm{E}[X_i]$,

$$\mathrm{E}[c^{\Delta_i}] \le c^{-\lambda} e^{\lambda(c-1)} = e^{\lambda(c-1-\ln c)}.$$

(d) Argue that $\mathrm{E}[c^{\Delta}] \le e^{\mathrm{E}[X](c-1-\ln c)}$ and complete the proof of the theorem.

[†]Recall that each X_i is a random variable whose value is in between 0 and 1.