CS5314 Randomized Algorithms

Homework 1

Due: 13:10 pm, October 14, 2008 (before class)

1. (10%) Suppose that we roll ten fair standard dice. What is the probability that their sum will be divisible by 6, assuming that the rolls are independent?

2. (10%) Suppose that Batman falls into Joker's pitfall. Joker sets up a time-bomb on Rachel Dawes' body, and there are three wires, blue, black, and red on this time-bomb. Two of these wires are connected to the alarm clock, and the remaining one is connected to the fuse. If *both* wires connecting to the alarm clock are cut, the alarm clock will stop. On the other hand, the time-bomb will explode if the wire connecting to the fuse is cut.

   Now, Joker asks Batman to pick one wire, so that Joker will cut this wire for Batman.

   After Batman has selected a wire, Joker will cut one of the unselected wires and show that it is connected to the alarm clock. (In case both unselected wires are connected to the alarm clock, Joker cuts one randomly.) After that, Joker asks Batman to decide whether he wants to stay with his first choice or to switch to the remaining unselected wire.

   Is it more advantageous for Batman to change his choice?

3. (20%) Let X and Y be independent geometric random variables, where X has parameter p and Y has parameter q.

   (a) What is the probability $\Pr(\min(X, Y) = k)$?

   (b) What is $\mathrm{E}[X \mid X \leq Y]$?

4. (10%) Let A be an array of n distinct numbers. An inversion in A is defined as a pair (i, j) such that $i < j$ and $A[i] > A[j]$. If the elements in A are arranged randomly, so that each of the $n!$ permutations is equally likely, what is the expected number of inversions?

5. (20%) (a) Alice and Bob decide to have children until either they have their first girl or they have $k \geq 1$ children. Assume that each child is a boy or girl independently with probability 1/2 and that there are no multiple births. What is the expected number of girls they have? What is the expected number of boys they have?

   (b) Suppose Alice and Bob simply decide to keep having children until they have their first girl. Assume that this is possible, what is the expected number of boys they have?

6. (10%) A permutation $\pi : [1, n] \to [1, n]$ can be represented as a set of cycles as follows. Let there be one vertex for each number i, $i = 1, \ldots, n$. If the permutation maps the number i to the number $\pi(i)$, then a directed arc is drawn from vertex i to vertex $\pi(i)$. This leads to a graph that is a set of disjoint cycles. Notice that some of the cycles could be self-loop. What is the expected number of cycles in a random permutation of n numbers?

7. (20%) You need a new staff assistant, and you have n people to interview. You want to hire the best candidate for this position. When you interview the candidates, you can give each of them a score, with the highest score will be the best and no ties being possible.

   You interview the candidates one by one. Because of your company's hiring policy, after you interview the kth candidate, you either offer the candidate the job immediately, or you will forever lose the chance to hire that candidate.

We suppose that the candidates are interviewed in a random order, chosen uniformly at random from all $n!$ possible orderings.

Consider the following strategy. First, interview m candidates but reject them all. Then from the $(m+1)$th candidate, hire the first one who is better than *all* of the previous candidates you have interviewed.[1]

(a) Let E be the event that we hire the best assistant, and let E_i be the event that ith candidate is the best and we hire him. Determine $\Pr(E_i)$, and show that

$$\Pr(E) = \frac{m}{n} \sum_{j=m+1}^{n} \frac{1}{j-1}.$$

(b) Bound $\sum_{j=m+1}^{n} \frac{1}{j-1}$ to obtain

$$\frac{m}{n}(\log_e n - \log_e m) \leq \Pr(E) \leq \frac{m}{n}(\log_e(n-1) - \log_e(m-1)).$$

(c) Show that $m(\log_e n - \log_e m)/n$ is maximized when $m = n/e$. Explain why this means $\Pr(E) \geq 1/e$ for this choice of m.

8. (Bonus, 10%) Consider adapting the min-cut algorithm to the problem of finding an s-t min-cut in an undirected graph. In this problem, we are given an undirected graph G together with two distinguished vertices s and t. An s-t min-cut is a set of edges whose removal disconnects s from t, and we seek an edge set of minimum cardinality. As the algorithm proceeds, the vertex s may get amalgamated into a new vertex as the result of an edge being contracted; we call this vertex the s-vertex. Similarly, we have the t-vertex. We run the algorithm as before, except that we reject moves that would contract an edge between the s-vertex and the t-vertex Show that there are simple graphs in which the probability that this algorithm finds an s-t min-cut is exponentially small.

[1] That is, you will hire the kth candidate if $k > m$ and this candidate is better than all of the $k-1$ candidates you have already interviewed.