1. (10%) A fixed point of a permutation $\pi : [1, n] \rightarrow [1, n]$ is a value for which $\pi(x) = x$. Find the variance in the number of fixed points of a permutation chosen uniformly at random from all permutations.

*Hint:* Let $X_i$ be an indicator such that $X_i = 1$ if $\pi(i) = i$. Then, $\sum_{i=1}^{n} X_i$ is the number of fixed points. You cannot use linearity to find $\text{Var}[\sum_{i=1}^{n} X_i]$, but you can calculate it directly.

2. (20%) The weak law of large numbers state that, if $X_1, X_2, X_3, \ldots$ are independent and identically distributed random variables with finite mean $\mu$ and finite standard deviation $\sigma$, then for any constant $\epsilon > 0$ we have

$$\lim_{n \to \infty} \Pr\left( \left| \frac{X_1 + X_2 + X_3 + \ldots + X_n}{n} - \mu \right| > \epsilon \right) = 0$$

Use Chebyshev's inequality to prove the weak law of large numbers.

3. (20%) Suppose you are given a biased coin that has $\Pr(\text{head}) = p$. Also, suppose that we know $p \geq a$, for some fixed $a$. Now, consider flipping the coin $n$ times and let $n_H$ be the number of times a head comes up. Naturally, we would estimate $p$ by the value $\tilde{p} = n_H/n$.

(a) Show that for any $\epsilon \in (0, 1),$

$$\Pr(|p - \tilde{p}| > \epsilon p) < \exp\left(\frac{-n a \epsilon^2}{2}\right) + \exp\left(\frac{-n a \epsilon^2}{3}\right)$$

(b) Show that for any $\delta \in (0, 1)$, if

$$n > \frac{2 \ln(2/\delta)}{a \epsilon^2},$$

then

$$\Pr(|p - \tilde{p}| > \epsilon p) < \delta.$$ 

4. (20%) Let $X_1, X_2, \ldots, X_n$ be independent Poisson trials such that $\Pr(X_i) = p_i$. Let $X = \sum_{i=1}^{n} X_i$ and $\mu = E[X]$. During the class, we have learnt that for any $\delta > 0,$

$$\Pr(X \geq (1 + \delta)\mu) < \left( \frac{e^\delta}{(1 + \delta)^{(1+\delta)}} \right)^\mu$$

In fact, the above inequality holds for the weighted sum of Poisson trials. Precisely, let $a_1, \ldots, a_n$ be real numbers in $[0, 1]$. Let $W = \sum_{i=1}^{n} a_i X_i$ and $\nu = E[W]$. Then, for any $\delta > 0,$

$$\Pr(W \geq (1 + \delta)\nu) < \left( \frac{e^\delta}{(1 + \delta)^{(1+\delta)}} \right)^\nu$$
(a) Show that the above bound is correct.
(b) Prove a similar bound for the probability \( \Pr(W \leq (1 - \delta)\nu) \) for any \( 0 < \delta < 1 \).

5. (30%) Consider a collection \( X_1, X_2, \ldots, X_n \) of \( n \) independent geometric random variables with parameter \( 1/2 \). Let \( X = \sum_{i=1}^{n} X_i \) and \( 0 < \delta < 1 \).

(a) By applying Chernoff bound to a sequence of \( (1 + \delta)(2n) \) fair coin tosses,\(^\dagger\) show that
\[
\Pr(X > (1 + \delta)(2n)) < \exp\left(\frac{-n\delta^2}{2(1 + \delta)}\right).
\]

(b) Derive a Chernoff bound on \( \Pr(X > (1+\delta)(2n)) \) using the moment generating function for geometric random variables as follows:

(i) Show that
\[
\mathbb{E}[e^{tX}] = \frac{e^t}{2 - e^t}.
\]

(ii) Show that
\[
\left|\frac{1}{(2 - e^t)e^{t(1+2\delta)}}\right|
\]

is minimized when \( t = \ln(1 + \delta/(1 + \delta)) \).

(iii) Show that
\[
\Pr(X > (1 + \delta)(2n)) < \left(\left(1 - \frac{\delta}{1 + \delta}\right)\left(1 + \frac{\delta}{1 + \delta}\right)^{1+2\delta}\right)^{-n}.
\]

(c) It is known that when \( \delta \) is small, there exists \( \varepsilon > 0 \) such that
\[
1 - \frac{\delta}{1 + \delta} > e^{-\varepsilon}, \quad \left(1 + \frac{\delta}{1 + \delta}\right)^{(1+\delta)/\delta} > e^{1-\varepsilon}, \quad \text{and} \quad \frac{(1 + 2\delta)\delta}{1 + \delta} > \delta^2.
\]

Show that in this case, the bound in 5(b)-(iii) becomes
\[
\Pr(X > (1 + \delta)(2n)) < \exp\left(-n(1 - \varepsilon)\delta^2 - \varepsilon\right).
\]

Conclude that when \( \delta \) is small enough such that \( \varepsilon \) is arbitrarily close to 0, the above bound is tighter than the bound obtained in 5(a).

\(^\dagger\)Here, we just assume \( (1 + \delta)(2n) \) is an integer.