CS5314 Randomized Algorithms

Tutorial 4 (Hint of Homework 3)

1. We prove that if Z is a Poisson random variable of mean μ, where $\mu \geq 1$ is an integer, then $\Pr(Z \geq \mu) \geq 1/2$ and $\Pr(Z \leq \mu) \geq 1/2$.

   (a) Show that $\Pr(Z = \mu + h) \geq \Pr(Z = \mu - h - 1)$ for $0 \leq h \leq \mu - 1$.

   (b) Using part (a), argue that $\Pr(Z \geq \mu) \geq 1/2$.

   Hint: Use the definition of Poisson distribution $\Pr(Z = j) = e^{-\mu}\mu^j/j!$.

2. In Page 15 of Lecture Notes 14, we showed that for any nonnegatvie function f,

$$E[f(Y_1^{(m)}, \cdots, Y_n^{(m)})] \geq E[f(X_1^{(m)}, \cdots, X_n^{(m)})]\Pr(\sum Y_i^{(m)} = m).$$

   (a) Now, suppose we further know that $E[f(X_1^{(m)}, \cdots, X_n^{(m)})]$ is monotonically increasing in m. Show that

$$E[f(Y_1^{(m)}, \cdots, Y_n^{(m)})] \geq E[f(X_1^{(m)}, \cdots, X_n^{(m)})]\Pr(\sum Y_i^{(m)} \geq m).$$

   (b) Make a similar statement for the case when $E[f(X_1^{(m)}, \cdots, X_n^{(m)})]$ is monotonically decresasing in m.

   (c) Combining part (a) and part (b) with the results in Question 1, prove the theorem in Page 20 of Lecture Notes 14.

   Hint: Check Page 15 of Lecture Notes 14.

3. Bloom filters can be used to estimate set differences. Suppose you have a set X and I have a set Y, both with n elements. For example, the sets might represent our 100 favorite songs. We both create Bloom filters of our sets, using the same number of bits m and the *same* k hash functions. Determine the expected number of bits where our Bloom filters differ as a function of m, n, k, and $|X \cap Y|$. **Hint:** Try the special cases when $|X \cap Y|$ is 0, 1, or 2. Obtain a general formula in terms of $|X \cap Y|$.

4. For the leader election problem briefly introduced in Lecture Notes 15, we have n users, each with an identifier. Suppose that we have a *good* hash function (that looks uniform and independent), which outputs a b-bit hash value for each identifier. One way to solve the leader election problem is as follows: Each user obtains the hash value from its identifier, and the leader is the user with the smallest hash value.

   Give lower and upper bounds on the number of bits b necessary to ensure that a unique leader is successfully chosen with probability p. Make your bounds as tight as possible.

   Hint: An upper bound of b can be obtained using similar method as in Lecture Notes 15 (Breaking Symmetry). For the lower bound, first derive an *exact* formula for the success probability in terms of b, and then simplify the formula.

5. (Further studies: No marks) Prove the theorem in Page 8 of Lecture Notes 16.

   Hint: Use Balls-and-Bins model to relate the the scenario of having isolated vertices in $G_{n,N}$ with the scenario of having empty bins when $2N$ balls are thrown in n bins. Contrast carefully the difference of the two scenarios, and show the bound.