Tutorial I
Outline

• Stable Marriage problem
• Assignment 1
Stable Marriage problem

- There are $N$ men (denoted by capital letters) and $N$ women (denoted by lower case letters).
- Each person has a preference list of the opposite sex.
- A marriage is 1-1 correspondence between men and women.
- An *unstable marriage* is that if there exist 2 marriage couples $X-x$ and $Y-y$ such that $X$ desires $y$ more than $x$ and $y$ desires $X$ more than $Y$. In this case, we say $X-y$ is dissatisfied.
Stable Marriage problem (con’t)

Ex. N=4

• men
  A:abcd  B:bacd  C:adcb  D:dcab
• women
  a:ABCD  b:DCBA  c:ABCD  d:CDAB
• There is a unstable marriage:
  A-a, B-b, C-c, D-d
• C-d is dissatisfied
Stable Marriage problem (con’t)

- Algorithm to stable marriage:
  - Men are numbered in some arbitrary order
  1. The lowest numbered unmarried man X proposes to the most desirable woman x on his list who has not rejected him
  2. The women x will accept the proposal if she is currently unmarried, or if her current mate Y is less desirable to her than X (Y reverts to the unmarried state)
  3. Repeats 1 and 2, terminating when every person has married
Stable Marriage problem (con’t)

• Another example:
  – A:abcd  B:bacd  C:adcb  D:dcab
  – a:ABCD  b:DCBA  c:ABCD  d:DCAB
  – Let the order of the men: ABCD
    • A proposes to a => A-a
    • B proposes to b => B-b
    • C proposes to a => a rejects C
    • C proposes to d => C-d
    • D proposes to d => d jilt C, D-d, C is single
    • C proposes to c => C-c
    • A-a, B-b, C-c, D-d
Stable Marriage problem (con’t)

• **Dose the algorithm terminate?**
  – Yes. At most $n^2$ proposals ... [why?]

• **The final marriage is stable?**
  – Yes. Proof by contradiction.
    1. Let X-y be a dissatisfied pair. In the marriage, they are paired as X-x, Y-y
    2. Since X prefers y to x, he must have proposed to y before getting married to x
    3. So, y either “rejected X” or “accepted X and then jilt him later”. This implies that her mate must be more desirable to her than X (contradiction)
Stable Marriage problem (con’t)

- Worst case of the algorithm: $O(n^2)$ proposals
- How to analysis the average case of the deterministic algorithm?
  - We assume that the men’s lists are chosen independently and uniformly; the women’s lists can be arbitrary but must be fixed in advance.
  - $T_p$ is the number of proposals made during the execution of the algorithm
  - The running time is proportional to $T_p$.
    $\Rightarrow$ difficult to analysis $T_p$
Stable Marriage problem (con’t)

• By deferred decisions:
  – The entire set of random choices (of the men’s lists) is not made in advance
  – At each step of the proposal, we fix only the random choices that must be revealed to the algorithm
Stable Marriage problem (cont’d)

• Original algorithm (with deferred decisions):
  When making a proposal, the man picks a random woman from those he has not proposed before
  – That is, sampling without replacement

• Simplified algorithm:
  When making a proposal, the man picks a random woman from all women
  – That is, sampling with replacement

• The outputs are the same
  – Reason: In simplified algorithm, once a man was rejected by a woman, if he propose to the same woman afterwards, he will still be rejected
Stable Marriage problem (con’t)

• $T_p$: the number of proposals made by the original algorithm
• $T_a$: the number of proposals made by the simplified algorithm
• $T_p > m \implies T_a \geq T_p > m$
• So, $\Pr(T_p > m) \leq \Pr(T_a > m)$ for all $m$
  – we can make use of the above observation to find an upper bound of $T_p$
Stable Marriage problem (con’t)

- $T_a$: the number of proposal made (each proposal is made uniformly and independently to one of $n$ women)
- The algorithm terminates once all women have received at least one proposal $\Rightarrow$ coupon collector’s problem
- Thus, expected number of proposals = $O(n \lg n)$
Assignment 1 – problem 1

• The proof of principle of inclusion-exclusion

\[ Pr \left( \bigcup_{i=1}^{n} E_i \right) = \sum_{i=1}^{n} Pr(E_i) - \sum_{i<j} Pr(E_i \cap E_j) + \sum_{i<j<k} Pr(E_i \cap E_j \cap E_k) \]

\[ - \cdots + (-1)^{\ell+1} \sum_{i_1<i_2<\cdots<i_\ell} Pr \left( \bigcap_{r=1}^{\ell} E_{i_r} \right) \]

\[ + \cdots + (-1)^n Pr \left( \bigcap_{i=1}^{n} E_i \right). \]

• Hint: by induction
Assignment 1 – problem 2

• the values of $F$ are stored in a lookup table, 1/5 of the lookup table entries are changed
• $F((x + y) \mod n) = (F(x) + F(y)) \mod m$
• Give input $z$, $F(z)$?
• Hint: If $F(z)$ is changed, you never get correct answer. You can use the above formula.
• Describe a randomized algorithm for finding an $r$-cut with minimum number of edges.

• Hint: $r$-cut is a general case of $2$-cut.
Assignment 1 – problem 4

• The expected number of fixed points \((\pi(x)=x)\) in permutation \(\pi\)

• Hint: Try to come up with a simple way to analyze the expected number
Assignment 1 – problem 5

• Interview problem:
  – First interview m candidates but reject them all
  – From the (m+1)th candidate onwards, hire the first candidate who is better than all of the previous candidates you have interviewed

• Hint: $E_i$ be the event that the $i$th candidate is the best and we hire him