An Interesting Equality for Sum of Reciprocals of the Squares

$$\sum_{k=1}^{\infty} 1/k^2 = \pi^2/6$$

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Overview

- Some History about the Sum
- Review: Maclaurin Series
- Euler's "Proof"
- Expanding $\sin^{-1} x$ (or $\arcsin x$)
- Choe's Proof
- Sources for Further Reading

Some History about the Sum

- Let I denote the sum $\sum_{k=1}^{\infty} 1/k^2$
- Jakob Bernoulli (1654–1705) proved that $I < 2$. Although he and his brother (Johann) tried very hard, they were not able to find the exact value of I
  - Jakob said, "if anybody has discovered the answer that makes us feel so defeated, please contact us, we will be very grateful."
- Leonard Euler (1707–1783) gave a "proof" that $I = \pi^2/6$ in 1734
  - In fact, he continued to produce the sum of reciprocals of the positive even powers
- Here, we will look at an alternative proof by Boo Rim Choe in 1987

Maclaurin Series

Theorem (Maclaurin Series)

The function, $f(x)$, can be expressed by

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(k)}(0)}{k!}x^k + \cdots$$

How to prove?

Given $f(x)$, can we find its constant term?
Can we find the coefficient of its x term?
In general, what should be the coefficient of its x^k term?

Euler's Proof

- Euler observed that the function

$$\sin x$$

  has roots at $x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$

- Next, he observed that the infinite product

$$x\left(1 - \frac{x^2}{(\pi)^2}\right)\left(1 - \frac{x^2}{(2\pi)^2}\right)\left(1 - \frac{x^2}{(3\pi)^2}\right)\cdots$$

  also has roots at $x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$

- Euler believed that these two functions are equivalent
- By Maclaurin series on $\sin x$, we find that the coefficient of the x^3 term $= -1/6$
- On the other hand, for the infinite product, the coefficient of the x^3 term $= -I/\pi^2 = -\sum_{k=1}^{\infty} 1/(k^2\pi^2)$
- Thus, Euler concluded that $I = \pi^2/6$

Expanding Inverse of Sine (1)

Fact

For $|x| < 1$,

$$\int \frac{1}{\sqrt{1-x^2}}\ dx = \sin^{-1} x + C.$$

How to prove?

Let $x = \sin y$. Then, we have

$$\begin{aligned}\int \frac{1}{\sqrt{1-x^2}}\ dx &= \int \frac{1}{\sqrt{1-\sin^2 y}}\ d(\sin y) \\ &= \int \frac{1}{\cos y} \cos y\ dy \\ &= \int 1\ dy = y + C = \sin^{-1} x + C.\end{aligned}$$

Expanding Inverse of Sine (2)

Fact

For $|x| < 1$,

$$\sin^{-1} x = \sum_{k=0}^{\infty} \frac{1 \cdot 3 \cdot \cdots \cdot (2k-1)}{2 \cdot 4 \cdot \cdots \cdot (2k)} \frac{x^{2k+1}}{2k+1}.$$

How to prove?

Directly follows from Maclaurin Series of $\sin^{-1} x$.

Expanding Inverse of Sine (3)

Corollary

For $|t| < \pi/2$,

$$t = \sum_{k=0}^{\infty} \frac{1 \cdot 3 \cdot \cdots \cdot (2k-1)}{2 \cdot 4 \cdot \cdots \cdot (2k)} \frac{\sin^{2k+1} t}{2k+1}.$$

How to prove?

Substituting $x = \sin t$ (with $|t| < \pi/2$ so that $t = \sin^{-1} x$) in the previous formula.

Choe's Proof (1)

Corollary

For $|t| < \pi/2$,

$$\int_0^{\pi/2} \sum_{k=0}^{\infty} \frac{1 \cdot 3 \cdot \cdots \cdot (2k-1)}{2 \cdot 4 \cdot \cdots \cdot (2k)} \frac{\sin^{2k+1} t}{2k+1} \; dt = \frac{\pi^2}{8}.$$

How to prove?

It follows since $\int_0^{\pi/2} t \; dt = \pi^2/8$.

Choe's Proof (2)

Fact

$$\int_0^{\pi/2} \sin^{2k+1} t \; dt = \frac{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2k)}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2k+1)}.$$

How to prove? (sketch)

Since

$$\int_0^{\pi/2} \sin^{2k+1} t \; dt = \int_0^1 (1 - y^2)^k \; dy,$$

we can use binomial expansion to show that the integral is equal to

$$\sum_{r=0}^{k} \binom{k}{r} \frac{(-1)^r}{(2r+1)}.$$

Then, the desired result follows by induction.

Choe's Proof (3)

Fact (Combining Everything)

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}.$$

Since

$$I - I/4 = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \qquad \text{(why?)},$$

we have:

Theorem

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

Sources for Further Reading

R. Chapman (2003).
Evaluating $\zeta(2)$. (This contains 14 proofs of the equality.)
http://secamlocal.ex.ac.uk/people/staff/rjchapma/etc/zeta2.pdf

B. R. Choe (1987).
An Elementary Proof of $\sum n^{-2} = \pi^2/6$,
American Mathematical Monthly, **volume 94, pages 662–663.**

Webpage: Summing Reciprocals.
http://library.thinkquest.org/28049/Summing%20reciprocals.html

Webpage: Pi Squared Over Six.
http://www.pisquaredoversix.force9.co.uk