1. Consider an instance of SAT with $m$ clauses, where every clause has exactly $k$ literals.

(a) Give a Las Vegas algorithm that finds an assignment satisfying at least $m(1 - 2^{-k})$ clauses, and analyze its expected running time.

**Ans.** Assign values independently and uniformly at random to the variables. The probability that $i$th clause with $k$ literals is satisfied is $(1 - 2^{-k})$. Let $N_c$ be the random variable indicating the number of satisfied clauses. Then

$$E[N_c] = \sum_{i=1}^{m} (1 - 2^{-k}) = m(1 - 2^{-k}).$$

Let $p = Pr(N_c \geq m(1 - 2^{-k}))$, and observe that $N_c \leq m$. Then we have:

$$m(1 - 2^{-k}) = E[N_c] = \sum_{i \leq m(1-2^{-k})} i \cdot Pr(N_c = i) + \sum_{i \geq m(1-2^{-k})} i \cdot Pr(N_c = i) \leq (1 - p)(m(1 - 2^{-k}) - 1) + pm,$$

which implies that

$$p \geq \frac{1}{1 + m \cdot 2^{-k}}.$$

So, the expected number of samples before finding an assignment satisfying at least $m(1 - 2^{-k})$ clauses is $1/p$, which is at most $1 + m \cdot 2^{-k}$. Testing to see if $(N_c \geq m(1 - 2^{-k}))$ can be done in $O(km)$ time. So, the algorithm can be done in polynomial time.

(b) Give a derandomization of the randomized algorithm using the method of conditional expectations.

**Ans.** Assign values to variables deterministically, one at a time, in an arbitrary order $x_1, x_2, \ldots, x_n$. Suppose that we have assigned the first $k$ variable. Let $y_1, y_2, \ldots, y_k$ be the corresponding assigned values. We compute the two quantities,

(i) $E[N_c \mid x_1 = y_1, x_2 = y_2, \ldots, x_k = y_k, x_{k+1} = T]$

(ii) $E[N_c \mid x_1 = y_1, x_2 = y_2, \ldots, x_k = y_k, x_{k+1} = F],$

and then choose the setting with larger expectation.

2. (a) Prove that, for every integer $n$, there exists a coloring of the edges of the complete graph $K_n$ by two colors so that the total number of monochromatic copies of $K_4$ is at most $\binom{n}{4}2^{-5}$.

**Ans.** $X$ is the random variable denoting the number of monochromatic copies of $K_4$. The probability that a certain 4-subset forms a monochromatic $K_4$ is $2 \cdot 2^{-6}$ (2 means two different colors.)

$$E[X] = \binom{n}{4} \cdot 2 \cdot 2^{-6} = \left(\frac{n}{4}\right)2^{-5}.$$
(b) Give a randomized algorithm for finding a coloring with at most \( \binom{n}{4}2^{-5} \) monochromatic copies of \( K_4 \) that runs in expected time polynomial in \( n \).

**Ans.** Color the edge independently and uniformly. Let \( p = \Pr(X \leq \binom{n}{4}2^{-5}) \). Then, we have

\[
\binom{n}{4}2^{-5} = E[X]
\]

\[
= \sum_{i \leq \binom{n}{4}2^{-5}} i \Pr(X = i) + \sum_{i > \binom{n}{4}2^{-5}+1} i \Pr(X = i)
\]

\[
\geq p + (1-p) \left( \binom{n}{4}2^{-5} + 1 \right),
\]

which implies that

\[
\frac{1}{p} \leq \binom{n}{4}2^{-5}.
\]

Thus, the expected number of samples is at most \( \binom{n}{4}2^{-5} \). Testing to see if \( X \leq \binom{n}{4}2^{-5} \) can be done in \( O(n^4) \) time. So, the algorithm can be done in polynomial time.

(c) Show how to construct such a coloring deterministically in polynomial time using the method of conditional expectations.

**Ans.** Similar to 1(b).

3. Given an \( n \)-vertex undirected graph \( G = (V,E) \), consider the following method of generating an independent set. Given a permutation \( \sigma \) of the vertices, define a subset \( S(\sigma) \) of the vertices as follows: for each vertex \( i, i \in S(\sigma) \) if and only if no neighbor \( j \) of \( i \) precedes \( i \) in the permutation \( \sigma \).

(a) Show that each \( S(\sigma) \) is an independent set in \( G \).

**Ans.** For any edge \((i,j)\), if \( i \) is in \( S(\sigma) \), it implies that \( (\sigma(i) > \sigma(j)) \). If \( j \) is in \( S(\sigma) \), it implies that \( (\sigma(j) > \sigma(i)) \). But, it is impossible that the two above cases occur at the same time. So, \( S(\sigma) \) is an independent set in \( G \).

(b) Suggest a natural randomized algorithm to produce \( \sigma \) for which you can show that the expected cardinality of \( S(\sigma) \) is

\[
\sum_{i=1}^{n} \frac{1}{d_i + 1}
\]

where \( d_i \) denotes the degree of vertex \( i \).

**Ans.** Get the permutation \( \sigma \) randomly (with respect to the uniform distribution). For any vertex \( i \), let \( U_i \) be the union of \( i \) and its neighbors. As the degree of \( i \) is \( d_i \), \( U_i \) has \( d_i + 1 \) elements. By the rule, \( i \) is in \( S(\sigma) \) if \( \sigma(i) \) is the smallest among \( \sigma(x) \) \( x \in U_i \). By symmetry, the probability of \( i \) in \( S(\sigma) \) is exactly \( 1/(d_i + 1) \). So, by linearity of expectation, the probability of \( i \) in \( S(\sigma) \) is

\[
E[|S(\sigma)|] = \sum_{i=1}^{n} \Pr(i \text{ is in } S(\sigma)) = \sum_{i=1}^{n} 1/(d_i + 1)
\]
(c) Prove that $G$ has an independent set of size at least $\sum_{i=1}^{n} 1/(d_i + 1)$.

**Proof.** By expectation argument, there must be at least one $S(\sigma)$ whose value is at least $E[|S(\sigma)|]$. And then, $S(\sigma)$ is an independent set in $G$. So, $G$ has an independent set of size at least $\sum_{i=1}^{n} 1/(d_i + 1)$.

4. We have shown using the probabilistic method that, if a graph $G$ has $n$ nodes and $m$ edges, then there exists a partition of the $n$ nodes into sets $A$ and $B$ such that at least $m/2$ edges cross the partition. Improve this result slightly: show that there exists a partition such that at least $mn/(2n - 1)$ edges cross the partition.

**Ans.** We choose $\lfloor n/2 \rfloor$ vertices uniformly from $n$ vertices to form a set $A$. Let $B$ be the set of remaining vertices. Now, we investigate the number of edges crossing the partition $(A, B)$ by two cases: $n$ is odd, and $n$ is even.

When $n$ is odd, set $n = 2k + 1$. The sizes of two sets $A$ and $B$ are $k$ and $k + 1$, respectively. For any edge $(u, v)$ in $G$, $\Pr((u, v) \text{ crosses the partition } (A, B))$ is:

\[
\begin{align*}
\frac{k}{2k+1} & \times \frac{k+1}{2k} \\
\frac{k+1}{2k+1} & \times \frac{k}{2k} \\
\text{ } & \text{ } \\
\geq & \frac{k+1}{2k+1} \frac{k}{2k} \\
\geq & \frac{n}{2n-1}.
\end{align*}
\]

When $n$ is even set $n = 2k$. The sizes of two sets $A$ and $B$ are both $k$. For any edge $(u, v)$ in $G$, $\Pr((u, v) \text{ crosses the partition } (A, B))$ is:

\[
\begin{align*}
\frac{k}{2k} & \times \frac{k}{2k-1} \\
\frac{k}{2k} & \times \frac{k}{2k-1} \\
\text{ } & \text{ } \\
= & \frac{k}{2k-1} \\
= & \frac{n}{2n-2} \\
\geq & \frac{n}{2n-1}.
\end{align*}
\]

Thus, in both cases, we can show that there exists a partition such that at least $mn/(2n - 1)$ edges cross the partition, using linearity of expectation.

5. Use the general form of the Lovasz local lemma to prove that the symmetric version of Theorem 6.11 can be improved by replacing the condition $4dp \leq 1$ by the weaker condition $ep(d+1) \leq 1$.

**Ans.** Set $x_i = 1/(d_i + 1)$. Then,

\[x_i \prod_{(i,j) \in E} (1 - x_j) = \left(1/(d+1) \right) \left(1 - 1/(d+1)\right)^d\]
\[
\begin{align*}
&\geq \ ep(1 - 1/(d + 1))^d \\
&= \ ep(d/(d + 1))^d \\
&= \ ep(1/d^d) \\
&\geq \ ep(1/e) \\
&= \ p,
\end{align*}
\]

which implies that

\[p \leq x_i \prod_{(i,j) \in E} (1 - x_j).\]

So,

\[\Pr(E_i) \leq p \leq x_i \prod_{(i,j) \in E} (1 - x_j).\]

By the general form of Lovasz local lemma, \textit{Q.E.D.}