CS5314 Randomized Algorithms

Homework 4
Due: June 14, 2007 (before class)

1. Consider an instance of SAT with m clauses, where every clause has exactly k literals.
   (a) Give a Las Vegas algorithm (i.e., an algorithm that always give a correct answer, but may vary in its running time) that finds an assignment satisfying at least $m(1-2^{-k})$ clauses, and analyze its expected running time.
   (b) Give a derandomization of the randomized algorithm using the method of conditional expectations.

2. (a) Prove that, for every integer n, there exists a coloring of the edges of the complete graph K_n by two colors so that the total number of monochromatic copies of K_4 is at most $\binom{n}{4}2^{-5}$.
   (b) (Further studies: No marks) Give a randomized algorithm for finding a coloring with at most $\binom{n}{4}2^{-5}$ monochromatic copies of K_4 that runs in expected time polynomial in n.
   (c) (Further studies: No marks) Show how to construct such a coloring deterministically in polynomial time using the method of conditional expectations.

3. Given an n-vertex undirected graph $G = (V, E)$, consider the following method of generating an independent set. Given a permutation σ of the vertices, define a subset $S(\sigma)$ of the vertices as follows: for each vertex i, $i \in S(\sigma)$ if and only if no neighbor j of i precedes i in the permutation σ.
   (a) Show that each $S(\sigma)$ is an independent set in G.
   (b) Suggest a natural randomized algorithm to produce σ for which you can show that the expected cardinality of $S(\sigma)$ is
   $$\sum_{i=1}^{n} \frac{1}{d_i + 1}$$
   where d_i denotes the degree of vertex i.
   (c) Prove that G has an independent set of size at least $\sum_{i=1}^{n} 1/(d_i + 1)$.

4. Use the general form of the Lovasz local lemma to prove that the symmetric version (theorem in Page 8 of Lecture Notes 20) can be improved by replacing the condition $4dp \leq 1$ by the weaker condition $ep(d + 1) \leq 1$.

5. (Further studies: No marks) We have shown using the probabilistic method that, if a graph G has n nodes and m edges, then there exists a partition of the n nodes into sets A and B such that at least $m/2$ edges cross the partition. Improve this result slightly: show that there exists a partition such that at least $mn/(2n - 1)$ edges cross the partition.