CS5314 Randomized Algorithms

Homework 3
Due: May 29, 2007 (before class)

1. We prove that if Z is a Poisson random variable of mean μ, where $\mu \geq 1$ is an integer, then $\Pr(Z \geq \mu) \geq 1/2$.

   (a) Show that $\Pr(Z = \mu + h) \geq \Pr(Z = \mu - h - 1)$ for $0 \leq h \leq \mu - 1$.

   (b) Using part (a), argue that $\Pr(Z \geq \mu) \geq 1/2$.

2. In Page 15 of Lecture Notes 14, we showed that for any nonnegatvie function f,

$$E[f(Y_1^{(m)}, \cdots, Y_n^{(m)})] \geq E[f(X_1^{(m)}, \cdots, X_n^{(m)})]\Pr(\sum Y_i^{(m)} = m).$$

   (a) Now, suppose we further know that $E[f(X_1^{(m)}, \cdots, X_n^{(m)})]$ is monotonically increasing in m. Show that

$$E[f(Y_1^{(m)}, \cdots, Y_n^{(m)})] \geq E[f(X_1^{(m)}, \cdots, X_n^{(m)})]\Pr(\sum Y_i^{(m)} \geq m).$$

   (b) Combining part (a) with the results in Question 1, prove the monotonically increasing case of the theorem in Page 20 of Lecture Notes 14.

3. Bloom filters can be used to estimate set differences. Suppose you have a set X and I have a set Y, both with n elements. For example, the sets might represent our 100 favorite songs. We both create Bloom filters of our sets, using the same number of bits m and the *same* k hash functions. Determine the expected number of bits where our Bloom filters differ as a function of m, n, k, and $|X \cap Y|$.

4. For the leader election problem briefly introduced in Lecture Notes 15, we have n users, each with an identifier. Suppose that we have a *good* hash function (that looks uniform and independent), which outputs a b-bit hash value for each identifier. One way to solve the leader election problem is as follows: Each user obtains the hash value from its identifier, and the leader is the user with the smallest hash value.

   Give lower and upper bounds on the number of bits b necessary to ensure that a unique leader is successfully chosen with probability p. Make your bounds as tight as possible.

5. (Further studies: No marks) Prove the theorem in Page 8 of Lecture Notes 16.