CS 2336
Discrete Mathematics

Lecture 15
Graphs: Planar Graphs
Outline

• What is a Planar Graph?
• Euler Planar Formula
  – Platonic Solids
  – Five Color Theorem
• Kuratowski’s Theorem
What is a Planar Graph?

Definition: A **planar graph** is an undirected graph that can be drawn on a plane without any edges crossing. Such a drawing is called a **planar representation** of the graph in the plane.

- Ex: $K_4$ is a planar graph
Examples of Planar Graphs

- Ex: Other planar representations of $K_4$
Examples of Planar Graphs

- Ex : $Q_3$ is a planar graph

![Diagram of a planar graph and its planar embedding]
Examples of Planar Graphs

• Ex: $K_{1,n}$ and $K_{2,n}$ are planar graphs for all $n$

$K_{1,5}$

$K_{2,4}$
Euler’s Planar Formula

Definition: A planar representation of a graph splits the plane into regions, where one of them has infinite area and is called the infinite region.

• Ex:

4 regions
(R₄ = infinite region)

2 regions
(R₂ = infinite region)
Euler’s Planar Formula

Let G be a connected planar graph, and consider a planar representation of G. Let

\[ V = \# \text{ vertices}, \ E = \# \text{ edges}, \ F = \# \text{ regions}. \]

**Theorem:** \[ V + F = E + 2. \]

**Ex:**

- \( V = 4, \ F = 4, \ E = 6 \)
- \( V = 8, \ F = 6, \ E = 12 \)
Euler’s Planar Formula

• Proof Idea :
  • Add edges one by one, so that in each step, the subgraph is always connected
  • Use induction to show that the formula is always satisfied for each subgraph
  • For the new edge that is added, it either joins :
    (1) two existing vertices \( \Rightarrow V \sim, F \uparrow \)
    (2) one existing + one new vertex \( \Rightarrow V \sim, F \uparrow \)
Euler’s Planar Formula

\[ V + F = E + 2 \]

Case 1

Case 2

\[ V \uparrow, F \sim \]

\[ V \sim, F \uparrow \]
Euler’s Planar Formula

• Let $G$ be a connected simple planar graph with $V = \# \text{ vertices}, E = \# \text{ edges}$.

Corollary: If $V \geq 3$, then $E \leq 3V - 6$.

• Proof: Each region is surrounded by at least 3 edges (how about the infinite region?)

$\Rightarrow 3F \leq \text{total edges} = 2E$

$\Rightarrow E + 2 = V + F \leq V + \frac{2E}{3}$

$\Rightarrow E \leq 3V - 6$
Euler’s Planar Formula

Theorem: $K_5$ and $K_{3,3}$ are non-planar.

• Proof:

(1) For $K_5$, $V = 5$ and $E = 10$
   \[ E > 3V - 6 \rightarrow \text{non-planar} \]

(2) For $K_{3,3}$, $V = 6$ and $E = 9$.
   \[ \text{If it is planar, each region is surrounded by at least 4 edges (why?)} \]
   \[ F \leq \left\lfloor \frac{2E}{4} \right\rfloor = 4 \]
   \[ V + F \leq 10 < E + 2 \rightarrow \text{non-planar} \]
Platonic Solids

Definition: A Platonic solid is a convex 3D shape that all faces are the same, and each face is a regular polygon.
Platonic Solids

Theorem: There are exactly 5 Platonic solids

• Proof:

Let \( n \) = # vertices of each polygon

\( m \) = degree of each vertex

For a platonic solid, we must have

\[
\begin{align*}
nF &= 2E \\
V m &= 2E
\end{align*}
\]
Platonic Solids

• Proof (continued):

By Euler’s planar formula,

\[ 2\frac{E}{m} + 2\frac{E}{n} = V + F = E + 2 \]

\[ \Rightarrow \quad 1/m + 1/n = 1/2 + 1/E \quad \ldots \ldots \text{(*)} \]

Also, we need to have

\[ n \geq 3 \quad \text{and} \quad m \geq 3 \quad \text{[from 3D shape]} \]

but one of them must be \( = 3 \) \quad \text{[from (\text{*})]}
Platonic Solids

• Proof (continued):

→ Either

(i) \( n = 3 \) (with \( m = 3, 4, \) or 5)

(ii) \( m = 3 \) (with \( n = 3, 4, \) or 5)
Map Coloring and Dual Graph

A Map M

Dual Graph of M
Map Coloring and Dual Graph

Observation: A proper color of $M$ ↔ A proper vertex color the dual graph

Proper coloring: Adjacent regions (or vertices) have to be colored in different colors.
Five Color Theorem

• Appel and Haken (1976) showed that every planar graph can be 4 colored
  (Proof is tedious, has 1955 cases and many subcases)

• Here, we shall show that:

  Theorem : Every planar graph can be 5 colored.

• The above theorem implies that every map can be 5 colored (as its dual is planar)
Five Color Theorem

• Proof :

We assume the graph has at least 5 vertices. Else, the theorem will immediately follow.

Next, in a planar graph, we see that there must be a vertex with degree at most 5. Else,

\[2E = \text{total degree} \geq 3V\]

which contradicts with the fact \( E \leq 3V - 6 \).
Five Color Theorem

• Proof (continued):

Let $v$ be a vertex whose degree is at most 5.

Now, assume inductively that all planar graphs with $n - 1$ vertices can be colored in 5 colors.

Thus if $v$ is removed, we can color the graph properly in 5 colors.

What if we add back $v$ to the graph now??
Five Color Theorem

• Proof (continued):

Case 1: Neighbors of $v$ uses at most 4 colors

there is a 5$^{th}$ color for $v$
Five Color Theorem

- Proof (continued):
  
  Case 2: Neighbors of $v$ uses up all 5 colors

Can we save 1 color, by coloring the yellow neighbor in blue?
Five Color Theorem

• Proof ("Case 2" continued):

Can we color the yellow neighbor in blue?

First, we check if the yellow neighbor can connect to the blue neighbor by a "switching" yellow-blue path.
Five Color Theorem

• Proof (“Case 2” continued):

Can we color the yellow neighbor in blue?

If not, we perform “switching” and thus save one color for $v$. 
Five Color Theorem

• Proof ("Case 2" continued):

Can we color the yellow neighbor in blue?

Else, they are connected

$\Rightarrow$ orange and green cannot be connected by "switching path"
Five Color Theorem

• Proof ("Case 2" continued):

We color the orange neighbor in green!

⇒ we can perform "switching" (orange and green) to save one color for $v$
Definition: A **subdivision** operation on an edge \{ u, v \} is to create a new vertex w, and replace the edge by two new edges \{ u, w \} and \{ w, v \}.

• Ex:

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![Diagram](image-url)
Kuratowski’s Theorem

Definition: Graphs G and H are **homeomorphic** if both can be obtained from the same graph by a sequence of subdivision operations.

- Ex: The following graphs are all homeomorphic:
Kuratowski’s Theorem

• In 1930, the Polish mathematician Kuratowski proved the following theorem:

Theorem:

Graph $G$ is non-planar if and only if $G$ has a subgraph homeomorphic to $K_5$ or $K_{3,3}$

• The “if” case is easy to show (how?)
• The “only if” case is hard (I don’t know either ...)
Kuratowski’s Theorem

• Ex: Show that the Petersen graph is non-planar.
Kuratowski’s Theorem

• Proof:

Petersen Graph  Subgraph homeomorphic to $K_{3,3}$
Kuratowski’s Theorem

• Ex: Is the following graph planar or non-planar?
Kuratowski’s Theorem

• Ans: Planar