CS 2336
Discrete Mathematics

Lecture 11
Sets, Functions, and Relations: Part III
Outline

• What is a Relation?
• Types of Binary Relations
• Representing Binary Relations
• Closures
Cartesian Product

• Let A and B be two sets

The cartesian product of A and B, denoted by \( A \times B \), is the set of all ordered pairs

\[
\{ (a, b) \mid a \in A \text{ and } b \in B \}
\]

• \( A = \{ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K \} = \text{all ranks} \)

\( B = \{ \spadesuit, \heartsuit, \diamondsuit, \clubsuit \} = \text{all suits} \)

\( A \times B = \text{all 52 cards in a deck} \)

\[
= \{ (1, \spadesuit), (2, \spadesuit), (3, \spadesuit), \ldots, (J, \clubsuit), (Q, \clubsuit), (K, \clubsuit) \}
\]
Cartesian Product

• Let $A_1, A_2, \ldots, A_k$ be $k$ sets

  The cartesian product of $A_1, A_2, \ldots, A_k$, denoted by $A_1 \times A_2 \times \cdots \times A_k$, is the set of all ordered pairs
  \[
  \{ (a_1, a_2, \ldots, a_k) \mid a_j \in A_j \text{ for all } j = 1, 2, \ldots, k \}
  \]

• Let $A_j = \text{the set } \mathbb{R} \text{ of real numbers, for all } j$
  $\Rightarrow$ $A_1 \times A_2 \times A_3 = \text{the 3-d Euclidean space } \mathbb{R}^3$
A binary relation from $A$ to $B$ is a subset of the cartesian product $A \times B$

- Example:
  
  $A = \{ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K \}$
  
  $B = \{ \spadesuit, \heartsuit, \diamondsuit, \clubsuit \}$
  
  $\text{Spades} = \{ (1, \spadesuit), (2, \spadesuit), (3, \spadesuit), \ldots, (Q, \spadesuit), (K, \spadesuit) \}$

$\Rightarrow$ Spades is a binary relation
A \textit{k-ary relation} is a subset of a cartesian of \textit{k} sets

\begin{itemize}
  \item Example:
  \[ L = \{ (x, y, z) \mid 2x + 3y + z = 0, \text{ and } x, y, z \in \mathbb{R} \} \]
  \implies \text{the line } L \text{ is a ternary relation of the space } \mathbb{R}^3
\end{itemize}
In the remaining of this lecture, we focus on a special type of relations: the binary relation from a set $A$ to $A$.

Example: $A =$ the set of integers $R = \{(a, b) \mid a - b \geq 10\}$
Types of Binary Relations

A binary relation \( R \) on \( A \) is said to be **reflexive** if \( (a, a) \in R \) for every \( a \in A \)

• Which of the following relations are reflexive?
  • \( R = \{ (a, b) \mid a - b \geq 10, \ a, b \text{ are integers} \} \)
  • \( S = \{ (a, b) \mid a \leq b, \ a, b \text{ are integers} \} \)
  • \( T = \{ (a, b) \mid a < b, \ a, b \text{ are integers} \} \)
  • \( U = \{ (x, y) \mid x \text{ and } y \text{ are on the same weekday, } x, y \text{ are days in April 2013} \} \)
Types of Binary Relations

A binary relation $R$ on $A$ is said to be \textit{symmetric} if $(a, b) \in R$ implies $(b, a) \in R$

• Which of the following relations are symmetric?
  
  • $R = \{ (a, b) \mid a - b \geq 10, \ a, b \text{ are integers} \}$
  
  • $S = \{ (a, b) \mid a \leq b, \ a, b \text{ are integers} \}$
  
  • $T = \{ (a, b) \mid a < b, \ a, b \text{ are integers} \}$
  
  • $U = \{ (x, y) \mid x \text{ and } y \text{ are on the same weekday, } x, y \text{ are days in April 2013} \}$
Types of Binary Relations

A binary relation $R$ on $A$ is said to be **antisymmetric** if $(a, b) \in R$ implies $(b, a) \notin R$ unless $a = b$

• Which of the following are antisymmetric?
  • $R = \{ (a, b) \mid a - b \geq 10, \ a, b \text{ are integers} \}$
  • $S = \{ (a, b) \mid a \leq b, \ a, b \text{ are integers} \}$
  • $T = \{ (a, b) \mid a < b, \ a, b \text{ are integers} \}$
  • $U = \{ (x, y) \mid x \text{ and } y \text{ are on the same weekday, } x, y \text{ are days in April 2013} \}$
A binary relation $R$ on $A$ is said to be **transitive** if $(a, b), (b, c) \in R$ implies $(a, c) \in R$

- Which of the following are transitive?
  - $R = \{ (a, b) \mid a - b \geq 10, \ a, b \ \text{are integers} \}$
  - $S = \{ (a, b) \mid a \leq b, \ a, b \ \text{are integers} \}$
  - $T = \{ (a, b) \mid a < b, \ a, b \ \text{are integers} \}$
  - $U = \{ (x, y) \mid x \ \text{and} \ y \ \text{are on the same weekday,} \ x, y \ \text{are days in April 2013} \}$
Representing Binary Relations

• Let A be a finite set

• The binary relation on A can be convenient represented in two different ways:

• Method 1: Matrix Form

  \[ A = \{ 1, 2, 3, 4 \} \]
  \[ R = \{ (1,1), (1,2), (2,3), (2,4), (3,4), (4,2) \} \]
Representing Binary Relations

- Method 2: Directed Graph
- $A = \{ 1, 2, 3, 4 \}$
- $R = \{ (1,1), (1,2), (2,3), (2,4), (3,4), (4,2) \}$
Closures

• Given a binary relation $R$, we may obtain a new relation $R'$ by adding items into $R$, such that $R'$ will have certain property

• Example:

  $R = \{ (1,1), (1,2), (2,3), (2,4), (3,4), (4,2) \}$

  If we add $(2,2)$, $(3,3)$, and $(4,4)$ into $R$, the resulting relation will be reflexive
Closures

- Let $R$ be a binary relation

The smallest possible relation $R'$ that contains $R$ as a subset, such that $R'$ has a property $P$, is the closure of $R$ with respect to $P$.
Closures

• What is the transitive closure of
R = { (1,1), (1,2), (2,3), (2,4), (3,4), (4,2) }?

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Transitive closure of R
Finding Transitive Closure

• Getting the transitive closure seems difficult
  Is there a systematic way to get this?

• Consider the directed graph representation
  \[ R = \text{all pairs of vertices where one can reach}
  \text{the other in 1 step} \]

• We can (not easily) show that for vertices \( x \) and \( y \),
  \[ x \text{ can reach } y \text{ in the graph} \iff (x, y) \text{ is in the transitive closure of } R \]
Finding Transitive Closure

Let \( R^k \) = all pairs of vertices where one can reach the other in exactly \( k \) steps

\[ R = R^1 \]

We can repeatedly obtain \( R^2, R^3, \) and so on, until we cannot add any new edges

\[ \text{the resulting graph corresponds to the transitive closure of } R \]
Finding Transitive Closure

Transitive closure of $R$
Finding Transitive Closure

• Apart from the previous method, there are faster ways to compute the transitive closure

• If the matrix form is given, and $|A| = n$
  1. Recursive doubling algorithm : $O(n^3 \log n)$ time
  2. Floyd-Warshall algorithm : $O(n^3)$ time

→ Code is super simple : Only 3 for loops!