CS 2336
Discrete Mathematics

Lecture 10
Sets, Functions, and Relations: Part II
Outline

- What is a Function?
- Types of Functions
- Floor and Ceiling Functions
- An Interesting Result
What is a Function?

• Suppose that each student in a mathematics class will be assigned a grade A, B, C, D, F

• Let us say the grades for the students are:
  Nobita (F), Shizuka (A), Takeshi (F), Suneo (B)
What is a Function?

• The previous is an example of a function

Let A and B be two nonempty sets

A function f from A to B is an assignment of exactly one item of B to each item of A.

This relationship is denoted by

\[ f : A \rightarrow B \]

We write \( f(a) = b \) if \( b \) is the unique item of B assigned by \( f \) to the item \( a \) of A, and we say \( b \) is the image of \( a \)

Terminology

• Function is also called **mapping** or **transformation**

• Given a function \( f : A \rightarrow B \)

  A is called the **domain**

  B is called the **codomain**

  The subset of B that contains all images,
  \[
  \{ b \mid f(a) = b \text{ for some } a \text{ in } A \},
  \]

  is called the **range**
Test Your Understanding

• What is the domain, codomain, and range in the above function?
Types of Functions

A function from A to B is said to be one-to-one, or injective, or an injection, if no two items of A have the same image in B.

• Which of the following are one-to-one functions?
  • \( f : \mathbb{Z} \rightarrow \mathbb{Z}, \) with \( f(x) = x^2 \)
  • \( g : \mathbb{N} \rightarrow \mathbb{Z}, \) with \( g(x) = x^2 \)
Types of Functions

A function from A to B is said to be **onto**, or **surjective**, or a **surjection**, if every item of B is the image of at least one item of A.

Equivalently, when range equals to codomain.

• Which of the following are onto functions?
  • $f : \mathbb{Z} \rightarrow \mathbb{Z}$, with $f(x) = 2x$
  • $g : \mathbb{R} \rightarrow \mathbb{R}$, with $g(x) = 2x$
Types of Functions

A function is said to be **one-to-one onto**, or **bijective**, or a **bijection**, if it is both one-to-one and onto.

- Which of the following are bijections?
  - \( f : \mathbb{Z} \to \mathbb{Z}, \) with \( f(x) = 2x \)
  - \( g : \mathbb{R} \to \mathbb{R}, \) with \( g(x) = 2x \)
Some Counting Problems

• Let $A$ and $B$ be two sets, with $|A| = m$, $|B| = n$

• Problems:
  1. How many distinct injections from $A$ to $B$?
  2. How many distinct surjections from $A$ to $B$?
  3. How many distinct bijections from $A$ to $B$?
Floor and Ceiling Functions

• Let $x$ be a real number

The **floor** function of $x$, denoted by $\lfloor x \rfloor$, is the largest integer that is smaller than or equal to $x$

The **ceiling** function of $x$, denoted by $\lceil x \rceil$, is the smallest integer that is larger than or equal to $x$

• Examples:

$\lfloor 0.5 \rfloor = 0$, $\lceil 0.5 \rceil = 1$, $\lfloor -1.1 \rfloor = -2$, $\lceil -1.1 \rceil = -1$

$\lfloor 7 \rfloor = 7$, $\lceil 7 \rceil = 7$, $\lfloor -4 \rfloor = -4$, $\lceil -4 \rceil = -4$
Floor and Ceiling Functions

• Some useful properties (n is an integer):

1. \( \lfloor x \rfloor = n \iff n \leq x < n + 1 \)
2. \( \lceil x \rceil = n \iff n - 1 < x \leq n \)
3. \( \lfloor x \rfloor = n \iff x - 1 < n \leq x \)
4. \( \lceil x \rceil = n \iff x \leq n < x + 1 \)
5. \( \lfloor -x \rfloor = -\lceil x \rceil \)
6. \( \lceil -x \rceil = -\lfloor x \rfloor \)
Challenges

• Which of the following are correct?
  1. \[\lfloor x + n \rfloor = \lfloor x \rfloor + n, \quad \text{where } n \text{ is an integer}\]
  2. \[\lceil x + y \rceil = \lceil x \rceil + \lceil y \rceil\]
  3. \[\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + 0.5 \rfloor\]

• Let \(z\) be a positive real. What is the maximum integer \(k\) such that
  \[\lfloor kz \rfloor \leq n,\]
  when \(n\) is a positive integer?
Answers to the Challenges

• 1 is correct.

Proof: Suppose \( \lfloor x \rfloor = m \), where \( m \) = integer.

\[ m \leq x < m + 1 \]

\[ m + n \leq x + n < m + n + 1 \]

\[ \lfloor x + n \rfloor = m + n = \lfloor x \rfloor + n \]

• 2 is wrong. Counter-example: \( x = 0.5, \ y = 0.5 \)
Answers to the Challenges

• 3 is correct.

Proof: Consider the value \( \{ x \} = x - \lfloor x \rfloor \). Then,

\[
\text{LHS} = \lfloor 2x \rfloor + 2 \{ x \} = 2 \lfloor x \rfloor + \lfloor 2 \{ x \} \rfloor
\]

There are two cases:

(i) \( 0 \leq \{ x \} < 0.5 \) \( \implies \) \( 0 \leq 2\{ x \} < 1 \)

\( \implies \) \( \text{LHS} = 2 \lfloor x \rfloor = \text{RHS} \)

(ii) \( 0.5 \leq \{ x \} < 1 \) \( \implies \) \( 1 \leq 2\{ x \} < 2 \)

\( \implies \) \( \text{LHS} = 2 \lfloor x \rfloor + 1 = \text{RHS} \)
Answers to the Challenges

• To find the maximum integer \( k \) with \( \lfloor kz \rfloor \leq n \), such a \( k \) must be the maximum integer with \( kz < n + 1 \), or \( k < (n + 1) / z \)

• There are two cases:
  (i) if \( (n + 1) / z \) is not an integer
      \[ k = \lfloor (n + 1) / z \rfloor = \lceil (n + 1) / z \rceil - 1 \]
  (ii) if \( (n + 1) / z \) is an integer
      \[ k = (n + 1) / z - 1 = \lceil (n + 1) / z \rceil - 1 \]

So it is always true that \( k = \lceil (n + 1) / z \rceil - 1 \)
An Interesting Result

• For a positive real number $z$, we define the spectrum of $z$,

$$\text{Spec}(z) = \{ \lfloor z \rfloor, \lfloor 2z \rfloor, \lfloor 3z \rfloor, \lfloor 4z \rfloor, \ldots \}$$

• Examples:

$$\text{Spec}(\sqrt{2}) = \{ 1, 2, 4, 5, 7, 8, 9, 11, 12, 14, \ldots \}$$

$$\text{Spec}(2+\sqrt{2}) = \{ 3, 6, 10, 13, 17, 20, \ldots \}$$

• Anything special about the above spectrums?
An Interesting Result

• Examples:

Let \( \phi = \left(1 + \sqrt{5}\right)/2 = 1.6180339887\ldots \)

\( = \) golden ratio

Then \( \phi^2 = \phi + 1 \)

\( = 2.6180339887\ldots \)

Spec(\( \phi \)) = \{1, 3, 4, 6, 8, 9, 11, 12, 14, 16, \ldots \}

Spec(\( \phi^2 \)) = \{2, 5, 7, 10, 13, 15, 18, \ldots \}

• Anything special about the above spectrums?
An Interesting Result

• In general, we have the following theorem:

Let $\alpha$ and $\beta$ be two positive irrational numbers such that

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1.$$ 

Then, the spectrums $\text{Spec}(\alpha)$ and $\text{Spec}(\beta)$ cover all the positive integers, and they have no common items.