CS 2336
Discrete Mathematics

Lecture 8
Counting: Permutations and Combinations
Outline

• Definitions
• Permutation
• Combination
• Interesting Identities
Definitions

• **Selection** and **arrangement** of objects appear in many places

  ➤ We often want to compute # of ways to select or arrange the objects

• Examples:
  1. How many ways to **select** 2 people from 5 candidates?
  2. How many ways to **arrange** 7 books on the bookshelf?
Definitions

• In most textbooks, we use the word 
  combination $\Leftrightarrow$ selection

An **r-combination of n objects** is an unordered 
selection of r objects from the n objects

• Example :
  
  \{ c, d \} is a 2-combination of \{ a, b, c, d, e \}
Definitions

• In most textbooks, we use the word permutation $\iff$ arrangement

An **r-permutation of n objects** is an ordered arrangement of r objects from the n objects

• Example:
  
  cabd is a 4-permutation of \{ a, b, c, d, e \}
Definitions

• Further, we define the following notation:

\[ C(n, r) \] denotes the number of \( r \)-combinations of \( n \) distinct objects

\[ P(n, r) \] denotes the number of \( r \)-permutations of \( n \) distinct objects

• What are the values of \( C(n, n) \), \( C(n, 1) \), \( C(3, 2) \), and \( P(3, 2) \) ?
Test Your Understanding

• Why are the following equalities correct?

1. \( P(n, r) = P(r, r) \times C(n, r) \)
2. \( P(n, n) = P(n, r) \times P(n - r, n - r) \)
3. \( C(n, r) = C(n, n - r) \)
Permutation

• In fact, there is a formula for $P(n, r)$:

$$P(n, r) = n (n - 1)(n - 2) \ldots (n - r + 1)$$

• Proof:

$P(n, r) = \# \text{ ways to get } r \text{ of } n \text{ objects in some order.}$

There are $n$ ways to choose the 1st object, $n - 1$ ways to choose the 2nd object, \ldots, $n - r + 1$ ways to choose the rth object

$\rightarrow$ Result follows from rule of product
Examples

• Ex 1: How many ways to select a first-prize, a second-prize, and a third-prize winners from 100 different people?

• Ex 2: How many ways can n people be ordered to form a ring?

The above are considered the same (as relative order is the same)
With Indistinguishable Objects

• How many different strings can be made by re-ordering the letters of the word “SUCCESS”?

• Answer:
  First, suppose that all the 7 letters are distinct. Then, there will be $7!$ different strings.
  Now, if we make the two Cs indistinguishable, we will only have $7!/2!$ different strings.
  Further, if the three Ss are indistinguishable, the number of different strings becomes $(7!/2!)/3!$.
With Indistinguishable Objects

• In general, if there are n objects, with
  \( n_1 \) indistinguishable objects of type 1,
  \( n_2 \) indistinguishable objects of type 2,
  … ,
  \( n_k \) indistinguishable objects of type k,

\[ \text{the number of n-permutations is:} \]

\[ \frac{n!}{n_1! \ n_2! \ \ldots \ n_k!} \]
Examples

• If we have 5 dashes and 8 dots, how many different ways to arrange them?

  . . . _ _ _ _ _ .

• If we can only use 7 symbols of them, how many different arrangements are there?

  . _ . _ . _ . _ _
Examples

• Show that for any positive integer $k$,

$$(k!)! \text{ is divisible by } k! \frac{(k-1)!}{k!}.$$ 

• For instance, when $k = 3$,

$$(k!)! = (3!)! = 6! = 720$$

$k! \frac{(k-1)!}{k!} = (3!)^2 = 6^2 = 36$
With Unlimited Repetitions

• Suppose that there are $n$ distinct objects, each with unlimited supply

• How many $r$-permutations are there? That is, how many ways to get a total of $r$ objects from them, and then form an arrangement?

• Answer: $n^r$
Examples

• Ex 1 : Consider all numbers between 1 and $10^{10}$
  (i) How many of them contain the digit 1 ?
  (ii) How many of them do not ?

• Ex 2 :
  (i) How many bit strings of length n are there ?
  (ii) How many contain even number of 0s?
Combination

• Recall that

\[ P(n, r) = P(r, r) \times C(n, r) \]

• Thus, we have

\[ C(n, r) = \frac{n (n - 1)(n - 2) \ldots (n - r + 1)}{r!} \]

• Alternatively, we can express \( C(n, r) \) as:

\[ C(n, r) = \frac{n!}{(n - r)! \ r!} \]
Examples

• Consider a hexagon where no three diagonals meet a one point

• How many diagonals are there?
• How many intersections between the diagonals?
• How many line segments are the diagonals divided by their intersections?
Examples

• In how many ways can we select 3 numbers from 1, 2, ..., 300, such that their sum is a multiple of 3?

• Hint:
  When the sum is a multiple of 3, what special property does the 3 numbers have?

• Answer: $100^3 + 3 \times C(100, 3)$
Examples

• Five pirates have discovered a treasure box
  They decided to keep the box in a locked room, so that all the locks of the room can be opened if and only if 3 or more pirates are present
• How to do so? How many locks do they need? (Each pirate may possess keys to different locks)
With Unlimited Repetitions

• Suppose that there are n distinct objects, each with unlimited supply

• How many r-combinations are there? That is, how many ways to get a total of r objects from them, and the ordering is not important?

• Answer: \( C(n - 1 + r, r) \) [Why?]
With Unlimited Repetitions

• Imagine we have a box for each type of objects

1 2 3 \ldots n

• A particular $r$-combination is equivalent to throwing a total of $r$ balls into these boxes

1 2 3 \ldots n
With Unlimited Repetitions

• To represent one of the r-combination, we may use a list of n – 1 bars and r stars, where
  ➢ the bars are used to mark off n different boxes
  ➢ the stars are used to indicate how many balls in each box

• For instance, suppose n = 5, r = 6

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1 2 3 4 5
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* * | * | | | * * *
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With Unlimited Repetitions

• Using the bars-and-stars representation, we see that
  ➢ each r-combination corresponds to a unique representation (with \( n - 1 \) bars and r stars), and
  ➢ each representation (with \( n - 1 \) bars and r stars) corresponds to a unique r-combination

⇒ \# of r-combinations = \# of representations

= \( C(n - 1 + r, r) \)
Examples

• Ex 1 : Suppose that a cookie shop has four different kinds of cookies. How many different ways can 6 cookies be chosen?

• Ex 2 : How many solutions does the equation

\[ x + y + z = 11 \]

have, if \( x, y, z \) are non-negative integers?

• Ex 3 : What if \( x, y, z \) are positive integers in Ex 2?
Interesting Identities

Pascal’s Identity:
\[ C(n, r) = C(n - 1, r) + C(n - 1, r - 1) \]

• Proof (by combinatorial arguments):
  To select \( r \) of \( n \) objects, there are in two cases:
  1. Get the first object, and then get \( r - 1 \) objects from the remaining \( n - 1 \) objects;
  2. Do not get the first object, and get \( r \) objects from the remaining \( n - 1 \) objects

\[ \Rightarrow \text{In total, } C(n - 1, r - 1) + C(n - 1, r) \text{ ways} \]
Interesting Identities

Binomial Theorem:

\[(x + y)^n = \sum_{r=0}^{n} C(n, r) x^{n-r} y^r\]

• Proof (by combinatorial arguments):
  The terms in \((x + y)^n\) must be of the form \(x^{n-r} y^r\).
  To obtain the term \(x^{n-r} y^r\), \(x\) is chosen \(n - r\) times from the \(n\) occurrences of \((x + y)\) in the product, so that \(y\) will be automatically chosen \(r\) times
  \(\Rightarrow\) the number of ways is exactly \(C(n, r)\)
Examples

• Ex 1 : What is the expansion of \((x + y)^4\) ?

• Ex 2 : What is the coefficient of \(x^{12}\) in \((2x - 3y)^{25}\) ?

• Ex 3 : What is the value of \(\sum_{r=0}^{n} C(n, r)\) ?

• Ex 4 : What is the value of \(\sum_{r=0}^{n} (-1)^r C(n, r)\) ?

• Ex 5 : What is the value of \(\sum_{r=0}^{n} 2^r C(n, r)\) ?
Interesting Identities

Vandermonde’s Identity :

\[ C(m + n, r) = \sum_{k=0}^{r} C(m, r - k) C(n, k) \]

• Proof (by combinatorial arguments):

To select \( r \) items from \( m + n \) distinct objects, we may assume that among these objects, \( m \) are white and \( n \) are black. The selection may start by selecting \( k \) black objects, and then the remaining from white objects. As \( k \) can vary from 0 to \( r \), this gives the result.
Example

• Can you simplify $\sum_{k=0}^{n} C(n, k)^2$?

• Answer:

Observe that

$$\sum_{k=0}^{n} C(n, k)^2 = \sum_{k=0}^{n} C(n, n-k) C(n, k)$$

By setting $m = n$ and $r = n$ in Vandermonde’s identity, we get the desired value as $C(2n, n)$