CS 2336
Discrete Mathematics

Lecture 3

Logic: Rules of Inference

Outline

- Mathematical Argument
- Rules of Inference

Argument

- In mathematics, an argument is a sequence of propositions (called premises) followed by a proposition (called conclusion)
- A valid argument is one that, if all its premises are true, then the conclusion is true
- Ex: "If it rains, I drive to school."

  "It rains."

  $\therefore$ "I drive to school."

Valid Argument Form

- In the previous example, the argument belongs to the following form:

$$\begin{array}{l} p \to q \\ p \\ \hline \therefore\ q \end{array}$$

- Indeed, the above form is valid no matter what propositions are substituted to the variables
- This is called a valid argument form

Valid Argument Form

- By definition, if a valid argument form consists
  - premises: $p_1, p_2, \ldots, p_k$
  - conclusion: q

  then $(p_1 \wedge p_2 \wedge \ldots \wedge p_k) \rightarrow q$ is a tautology
- Ex: $((p \rightarrow q) \wedge p) \rightarrow q$ is a tautology
- Some simple valid argument forms, called rules of inference, are derived and can be used to construct complicated argument form

Rules of Inference

1. Modus Ponens (method of affirming)

   premises: $p, \quad p \to q$

   conclusion: q

2. Modus Tollens (method of denying)

   premises: $\neg q, \quad p \to q$

   conclusion: $\neg p$

Rules of Inference

3. Hypothetical Syllogism

   premises: $p \to q$, $q \to r$

   conclusion: $p \to r$

4. Disjunctive Syllogism

   premises: $\neg p$, $p \vee q$

   conclusion: q

Rules of Inference

5. Addition

   premises: p

   conclusion: $p \vee q$

6. Simplification

   premises: $p \wedge q$

   conclusion: p

Rules of Inference

7. Conjunction

   premises: $p, \quad q$

   conclusion: $p \land q$

8. Resolution

   premises: $p \lor q, \quad \neg p \lor r$

   conclusion: $q \lor r$

Rules of Inference with Quantifiers

9. Universal Instantiation

   premises: $\forall x$ P(x)

   conclusion: P(c), for any c

10. Universal Generalization

    premises: P(c) for any arbitrary c

    conclusion: $\forall x$ P(x)

Rules of Inference with Quantifiers

11. Existential Instantiation

    premises: $\exists x\ P(x)$

    conclusion: P(c), for some element c

12. Existential Generalization

    premises: P(c) for some element c

    conclusion: $\exists x\ P(x)$

Applying Rules of Inferences

- Example 1: It is known that
  1. It is not sunny this afternoon, and it is colder than yesterday.
  2. We will go swimming only if it is sunny.
  3. If we do not go swimming, we will play basketball.
  4. If we play basketball, we will go home early.
- Can you conclude “we will go home early”?

Solution

- To simplify the discussion, let

  $p :=$ It is sunny this afternoon

  $q :=$ It is colder than yesterday

  $r :=$ We will go swimming

  $s :=$ We will play basketball

  $t :=$ We will go home early

- We will give a valid argument with

  premises: $\neg p \wedge r,\quad r \to p,\quad \neg r \to s,\quad s \to t$

  conclusion: t

Solution

Step	**Reason**
1. $\neg p \wedge r$	Premise
2. $\neg p$	Simplification using (1)
3. $r \rightarrow p$	Premise
4. $\neg r$	Modus Tollens using (2) and (3)
5. $\neg r \rightarrow s$	Premise
6. s	Modus Ponens using (4) and (5)
7. $s \rightarrow t$	Premise
8. t	Modus Ponens using (6) and (7)

Applying Rules of Inferences

- Example 2: It is known that
  1. If you send me an email, then I will finish my program.
  2. If you do not send me an email, then I will go to sleep early.
  3. If I go to sleep early, I will wake up refreshed.
- Can you conclude “If I do not finish my program, then I will wake up refreshed”?

Solution

- To simplify the discussion, let

  $p :=$ You send me an email

  $q :=$ I finish my program

  $r :=$ I go to sleep early

  $s :=$ I wake up refreshed

- We will give a valid argument with

  premises: $p \to q, \quad \neg p \to r, \quad r \to s$

  conclusion: $\neg q \to s$

Solution

Step	**Reason**
1. $p \rightarrow q$	Premise
2. $\neg q \rightarrow \neg p$	Contrapositive of (1)
3. $\neg p \rightarrow r$	Premise
4. $\neg q \rightarrow r$	Hypothetical Syllogism by (2) and (3)
5. $r \rightarrow s$	Premise
6. $\neg q \rightarrow s$	Hypothetical Syllogism by (4) and (5)

Applying Rules of Inferences

- Example 3: It is known that
  1. A student in this class has not read the book.
  2. Everyone in this class passed the first exam.

- Can you conclude that “Someone who passed the first exam has not read the book”?

Solution

- To simplify the discussion, let

  C(x) := x is a student in the class

  B(x) := x has read the book

  P(x) := x passed the first exam

- We will give a valid argument with

  premises: $\exists x\,(\,\mathrm{C}(x) \wedge \neg \mathrm{B}(x)\,)$,

  $\forall x\,(\,\mathrm{C}(x) \to \mathrm{P}(x)\,)$

  conclusion: $\exists x\,(\,\mathrm{P}(x) \wedge \neg\, \mathrm{B}(x)\,)$

Solution

Step	**Reason**
1. $\exists x\,(\,C(x) \wedge \neg B(x)\,)$	Premise
2. $C(a) \wedge \neg B(a)$	Existential Instantiation
3. $C(a)$	Simplification by (2)
4. $\forall x\,(\,C(x) \rightarrow P(x)\,)$	Premise
5. $C(a) \rightarrow P(a)$	Universal Instantiation
6. $P(a)$	Modus Ponens by (3) and (5)
7. $\neg B(a)$	Simplification by (2)
8. $P(a) \wedge \neg B(a)$	Conjunction by (6) and (7)
9. $\exists x\,(\,P(x) \wedge \neg\, B(x)\,)$	Existential Generalization

From Sherlock Holmes

- The following is from Silver Blaze, one of Sherlock Holmes stories (written by Sir Arthur Conan Doyle):

Gregory:	Is there any other point to which you would wish to draw my attention?
Holmes:	To the curious incident of the dog in the night-time.
Gregory:	The dog did nothing in the night-time.
Holmes:	That was the curious incident.