CS 2336
Discrete Mathematics

Lecture 2

Logic: Predicate Calculus

Outline

- Predicates
- Quantifiers
- Binding
- Applications
- Logical Equivalences

Predicates

- In mathematics arguments, we will often see sentences containing variables, such as:
  - $x > 0$
  - $x = y + 3$
  - Computer x is functioning properly
- For each sentence, we call
  - the variables as the subject of the sentence
  - the other part, which describes the property of the variables, as the predicate of the sentence

Predicates

- Take $x > 0$ as an example
  - x : the subject
  - > 0: the predicate
- Once the value of x is assigned, the above sentence becomes a proposition and has a truth value
- We can denote it as some function P(x) of x
  - P is called a propositional function

Predicates

- Example 1:
  Let P(x) denote the sentence "$x > 0$".
  What are the truth values of P(0) and P(1)?

- Example 2:
  Let Q(x, y) denote the sentence "$x = y + 3$".
  What are the truth values of Q(1,2) and Q(3,0)?

Quantifiers

- In English, the words all, some, many, none, few are used to express some property (predicate) is true over a range of subjects
  - These words are called quantifiers

- In mathematics, two important quantifiers are commonly used to create a proposition from a propositional function:

  universal quantifier and existential quantifier

Universal Quantifier

- Many mathematical statements say that a property is true for all values of a variable, when values are chosen from some domain
- Examples:
  - $z(z+1)(z+2)$ is divisible by 6 for all integer z
  - q^2 is rational for all rational number q
  - $r^3 > 0$ for all positive real number r
- Important Note: Domain needs to be specified!

Universal Quantifier

- Universal Quantifier

  The universal quantification of P(x) is the proposition

  "P(x) for all values of x in the domain."

  The notation $\forall x$ P(x) represents the above proposition.

  A value of x making the proposition false is called a counter-example.

Universal Quantifier

- If all values in the domain can be listed, say x_1, x_2, ..., x_k, then $\forall x\, P(x)$ is the same as

$$P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_k)$$

- Example:

  What is the truth value of $\forall x\, (x \leq 10)$ when the domain consists of all positive integers not exceeding 3?

  What is the truth value of $P(1) \wedge P(2) \wedge P(3)$?

Test Your Understanding

- What is the truth value of

$$\forall x\,(x^2 \geq x)$$

  - if the domain consists of all real numbers?
  - if the domain consists of all integers?

Test Your Understanding (Solution)

- False, if the domain consists of all real numbers. In particular, the case

$$x = 0.5$$

is a counter-example.

- True, if the domain consists of all integers. To see this, we notice the following equivalences:

$$x^2 \geq x \Leftrightarrow x\,(x-1) \geq 0 \Leftrightarrow x \leq 0 \text{ or } x \geq 1$$

Thus, $x^2 \geq x$ cannot be false, since there are no integers x with $0 < x < 1$.

Existential Quantifier

- Many mathematical statements say that a property is true for some value of a variable, when values are chosen from some domain
- Examples:
  - $2^{2^z} + 1$ is a prime for some non-negative integer z
  - r^s is rational for some irrational numbers r and s

- Important Note: Domain needs to be specified!

Existential Quantifier

- Existential Quantifier

  The existential quantification of P(x) is the proposition

  "P(x) for some value of x in the domain."

  The notation $\exists x$ P(x) represents the above proposition.

  The proposition is false if and only if P(x) is false for all values of x.

Existential Quantifier

- If all values in the domain can be listed, say x_1, x_2, …, x_k, then $\exists x\, P(x)$ is the same as

$$P(x_1) \vee P(x_2) \vee \ldots \vee P(x_k)$$

- Example:

  What is the truth value of $\exists x\, (x \leq 0)$ when the domain consists of all positive integers not exceeding 3?

  What is the truth value of $P(1) \vee P(2) \vee P(3)$?

Test Your Understanding

- What is the truth value of

$$\exists z\,(z^2 \geq 10)$$

  - if the domain consists of all positive integers not exceeding 3?
  - if the domain consists of all integers not exceeding 3?

Quantifiers with Restricted Domain

- Sometimes, we want to simplify the writing by using short-hand notation
- Assuming the domain consists of all integers, guess what does each of the following mean?
  - $\forall x < 0\ (x^2 > 0)$
  - $\forall y \neq 0\ (y^3 \neq 0)$
  - $\exists z > 0\ (z^2 = 10)$

Quantifiers with Restricted Domain

- $\forall x < 0\ (x^2 > 0\ )$ means

  "For every x in the domain with $x < 0$, $x^2 > 0$."

  The proposition is the same as:

  $$\forall x\ (x < 0 \rightarrow x^2 > 0\ )$$

- $\exists z > 0\ (z^2 = 10\ )$ means

  "There is some z in the domain with $z > 0$, $z^2 = 10$."

  The proposition is the same as:

  $$\exists z\ (z > 0 \wedge z^2 = 10\ )$$

Binding Variables

- If there is a quantifier used on a variable x, we say the variable is bound. Else it is free.
  - Ex: In $\exists x\,(x + y = 1)$, x is bound and y is free

- If all variables in a propositional function are bound, the function becomes a proposition
  - Ex: $\forall y\, \exists x\,(x + y = 1)$ is a proposition

Multiple Quantifiers

- In the last example, we have a proposition

$$\forall y \, \exists x \, (x + y = 1)$$

  with two quantifiers, where

  $\forall y$ is applied to $\exists x \, (x + y = 1)$, and

  $\exists x$ is applied to $x + y = 1$

- The part of the logical expression where a quantifier is applied is called the scope of that quantifier

Multiple Quantifiers

- How about this?

$$\forall y \neq 0\,(y^3 \neq 0) \wedge \exists x\,(x = 1)$$

What is the scope of y ?

- Quantifier is assumed to have a higher precedence than logical operators, so the above is the same as:

$$\forall x \neq 0\,(x^3 \neq 0) \wedge \exists x\,(x = 1)$$

The Order of Quantifiers

- Order in which quantifiers appear is important
- Example:

  Suppose that the domain for both x and y are integers. What are the truth values of the following?

  1. $\forall y \, \exists x \, (x + y = 1)$
  2. $\exists x \, \forall y \, (x + y = 1)$

The Order of Quantifiers

- Two special cases where the order of quantifiers is not important are:
  1. All quantifiers are universal quantifiers
  2. All quantifiers are existential quantifiers

- Example:

$$\exists\, x\ \exists\, y\ (x + y = 1)$$

  means the same as

$$\exists\, y\ \exists\, x\ (x + y = 1)$$

Applications: English Translation

- How to translate the following sentence

  "Every student in this class has studied Calculus."

  into a logical expression, if

  Q(x) denotes "x has studied Calculus", and
  the domain of x is all students in this class?

- What if the domain of x consists of all students in NTHU?

Applications: English Translation

- How to translate the following sentences
  1. "All lions are fierce."
  2. "Some lion does not drink coffee."
  3. "Some fierce creatures do not drink coffee."

  into logical expressions, if

  P(x) := "x is a lion", Q(x) := "x is fierce",
  R(x) := "x drinks coffee",
  and the domain of x consists of all creatures?

Applications: English Translation

- How to translate the following sentence

  "If a person is a female and is a parent, then this person is someone's mother"

  into a logical expression, if

  F(x) := "x is a female", P(x) := "x is a parent",

  M(x, y) := "x is a mother of y",

  and the domain consists of all people?

Applications: English Translation

- How to translate the following sentence

  "Every person has exactly one best friend"

  into a logical expression, if

  B(x, y) := "y is a best friend of x",

  and the domain consists of all people?

Applications: English Translation

- How to translate the following sentence

  "There is a woman who has taken a flight on every airline in the world"

  into a logical expression, if

  P(w, f) := "w has taken a flight f",
  Q(f, a) := "f is a flight of airline a",

  and the domains of w, f, a consist of all women in the world, all airplane flights, and all airlines, respectively?

Applications: Math Translation

- How to translate the statements

  1. “The sum of two positive integers is always positive”
  2. “Every real number, except 0, can find some real number such that their product is 1”

  into logical expressions?

Applications: Translating Expression

- How to translate the following expression

$$\exists x \, \forall y \, \forall z$$
$$((\, \mathrm{F}(x, y) \wedge \mathrm{F}(x, z) \wedge (y \neq z)) \rightarrow \neg\, \mathrm{F}(y, z)\,)$$

into English, if

$\mathrm{F}(x, y)$:= “x is a friend of y”,

and the domain consists of all people?

Logical Equivalences

- As in the case of propositions, some common logical equivalences have been derived for predicates and quantifiers

- Examples:
  1. $\exists x \, \exists y \; P(x, y) \equiv \exists y \, \exists x \; P(x, y)$
  2. $\forall x \, \forall y \; P(x, y) \equiv \forall y \, \forall x \; P(x, y)$

  hold for any predicate P, and any domain

Logical Equivalences

- De Morgan's Laws
  1. $\neg \forall x \, P(x) \equiv \exists x \neg P(x)$
  2. $\neg \exists x \, P(x) \equiv \forall x \neg P(x)$

- Can you show that

$$\neg \forall x \, (P(x) \rightarrow Q(x))$$

is equivalent to

$$\exists x \, (P(x) \wedge \neg Q(x)) \, ?$$