CS2336 Discrete Mathematics

Homework 2
Tutorial: April 10, 2014
Exam 1: April 14, 2014 (2 hours)

Problems marked with * will be explained in the tutorial.

1. (*) Give a direct proof for the following theorem: If n is perfect square, then $n + 2$ is not a perfect square.

2. (*) Use a direct proof to show that any odd integer is the difference of two squares.

3. Prove that for all real numbers x and y, if $x + y \geq 100$, then $x \geq 50$ or $y \geq 50$.

4. Show that for any real number x, $x^2 - 3x + 2 > 0$ if and only if $x < 1$ or $x > 2$.

5. For each of the following statements, provide an indirect proof by stating and proving the contrapositive of the given statement.

   (a) (*) For all integers m and n, if mn is odd, then m, n are both odd.

   (b) For all integers m and n, if $m + n$ is even, then m, n are both even or both odd.

6. (*) Use "prove by cases" to show the following results:

   (a) If n is a natural number, then $n^2 + n + 3$ is odd.

   (b) If a and b are real numbers, $|a - b| = |b - a|$

7. (*) Show that $x^5 - x^4 + x^3 - x^2 + x - 1 = 0$ has an integral root.

8. (*, Challenging without the hint) Prove that when a white square and a black square are removed from an 8×8 chessboard, you can tile the remaining squares of the checkerboard using dominoes.

   Hint: It is a fun problem! Try it without the hint. See Figure 2 only if you get stuck.

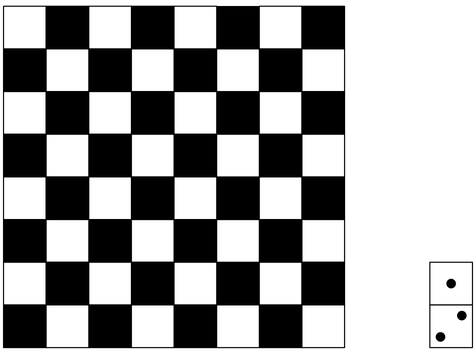

Figure 1: A checkerboard and a domino piece.

9. (*, Challenging) Let α be an angle such that $\alpha = \tan^{-1}(1/3) + \tan^{-1}(1/2)$ and $0 \leq \alpha < 2\pi$. Show that $\alpha = \pi/4$ without using a calculator.

10. (*) Prove or disprove the following:

If $p_1, p_2, \ldots, p_n$ are the n smallest primes, then $k = p_1 p_2 \cdots p_{n+1} + 1$ is prime.

11. Prove each of the following for all integer $n \geq 1$ by mathematical induction.

    (a) (*)
$$1 \times 3 + 2 \times 4 + 3 \times 5 + \cdots + n(n+2) = \frac{n(n+1)(2n+7)}{6}.$$

    (b)
$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}.$$

12. (*) Use strong induction to prove that $\sqrt{2}$ is irrational.
    Hint: Let $P(n)$ be the statement that $\sqrt{2} \neq n/b$ for any positive integer b.

13. Show that if any 14 integers are selected from the set $S = \{1, 2, 3, \ldots, 25\}$, there are at least two selected integers whose sum is 26.

14. (*) If 11 integers are selected from $\{1, 2, 3, \ldots, 100\}$, prove that there are at least two, say x and y, such that $0 < |\sqrt{x} - \sqrt{y}| < 1$.

15. (*, Challenging) Let $(a_1, a_2, a_3, a_4, a_5, a_6)$ and $(b_1, b_2, b_3, b_4, b_5, b_6)$ be two arrangements of the integers $1, 2, 3, 4, 5, 6$. Consider the six pairs of differences $|a_i - b_i|$. Is it possible that all of these differences are not the same?

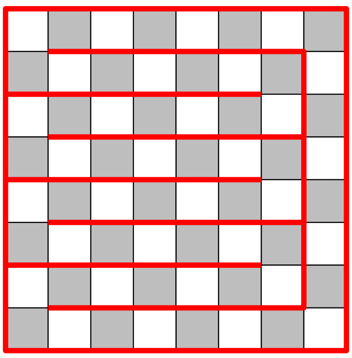

Figure 2: A hint for Question 8.