1. (*) Give a direct proof for the following theorem: If \( n \) is perfect square, then \( n + 2 \) is not a perfect square.

2. (*) Use a direct proof to show that any odd integer is the difference of two squares.

3. Prove that for all real numbers \( x \) and \( y \), if \( x + y \geq 100 \), then \( x \geq 50 \) or \( y \geq 50 \).

4. Show that for any real number \( x \), \( x^2 - 3x + 2 > 0 \) if and only if \( x < 1 \) or \( x > 2 \).

5. For each of the following statements, provide an indirect proof by stating and proving the contrapositive of the given statement.
   
   (a) (*) For all integers \( m \) and \( n \), if \( mn \) is odd, then \( m, n \) are both odd.
   
   (b) For all integers \( m \) and \( n \), if \( m + n \) is even, then \( m, n \) are both even or both odd.

6. (*) Use “prove by cases” to show the following results:
   
   (a) If \( n \) is a natural number, then \( n^2 + n + 3 \) is odd.
   
   (b) If \( a \) and \( b \) are real numbers, \( |a - b| = |b - a| \)

7. (*) Show that \( x^5 - x^4 + x^3 - x^2 + x - 1 = 0 \) has an integral root.

8. (*, Challenging without the hint) Prove that when a white square and a black square are removed from an \( 8 \times 8 \) chessboard, you can tile the remaining squares of the checkerboard using dominoes.
   
   \textit{Hint:} It is a fun problem! Try it without the hint. See Figure 2 only if you get stuck.

9. (*, Challenging) Let \( \alpha \) be an angle such that \( \alpha = \tan^{-1}(1/3) + \tan^{-1}(1/2) \) and \( 0 \leq \alpha < 2\pi \).
   
   Show that \( \alpha = \pi/4 \) without using a calculator.
10. (*) Prove or disprove the following:

If \( p_1, p_2, \ldots, p_n \) are the \( n \) smallest primes, then \( k = p_1 p_2 \cdots p_{n+1} + 1 \) is prime.

11. Prove each of the following for all integer \( n \geq 1 \) by mathematical induction.

(a) (*)

\[
1 \times 3 + 2 \times 4 + 3 \times 5 + \cdots + n(n + 2) = \frac{n(n + 1)(2n + 7)}{6}.
\]

(b)

\[
\sum_{i=1}^{n} \frac{1}{i(i + 1)} = \frac{n}{n + 1}.
\]

12. (*) Use strong induction to prove that \( \sqrt{2} \) is irrational.

\textit{Hint:} Let \( P(n) \) be the statement that \( \sqrt{2} \neq n/b \) for any positive integer \( b \).

13. Show that if any 14 integers are selected from the set \( S = \{1, 2, 3, \ldots, 25\} \), there are at least two selected integers whose sum is 26.

14. (*) If 11 integers are selected from \( \{1, 2, 3, \ldots, 100\} \), prove that there are at least two, say \( x \) and \( y \), such that \( 0 < |\sqrt{x} - \sqrt{y}| < 1 \).

15. (*) Challenging) Let \( (a_1, a_2, a_3, a_4, a_5, a_6) \) and \( (b_1, b_2, b_3, b_4, b_5, b_6) \) be two arrangements of the integers 1, 2, 3, 4, 5, 6. Consider the six pairs of differences \( |a_i - b_i| \). Is it possible that all of these differences are not the same?

Figure 2: A hint for Question 8.