CS5371
Theory of Computation

Lecture 23: Complexity VIII
(Space Complexity)

Objectives

- Introduce Space Complexity
- Savitch's Theorem
- The class PSPACE

Space Complexity

Definition [for DTM]: Let M be a DTM that halts on all inputs. The space complexity of M is a function $f: N \rightarrow N$, where $f(n)$ is the maximum number of tape cells that M scans on any input of length n.

If the space complexity of M is $f(n)$, we say M runs in space $f(n)$

Space Complexity (2)

Definition [for NTM]: Let M be an NTM that all branches halt on all inputs. The **space complexity** of M, $f(n)$, will be the maximum number of tape cells that M scans on any branch of its computation for any input of length n.

Again, if the space complexity of M is $f(n)$, we say M runs in space $f(n)$

Space Complexity Classes

Definition: Let $f: N \rightarrow R$ be a function. We define two notation for describing space complexity classes as follows:

SPACE($f(n)$) = { L | L is a language decided by a **DTM** M that runs in $f(n)$ space }

NSPACE($f(n)$) = { L | L is a language decided by an **NTM** M that runs in $f(n)$ space }

Example 1

Theorem: SAT is in SPACE(n)

Proof: The following DTM M decides SAT:

M = "On input ⟨F⟩,

1. For each truth assignment,
   (a) Evaluate F on that truth assignment
2. If F is evaluated to TRUE in some case, accept. Otherwise, reject."

The space usage is O(length of ⟨F⟩). Why??

Example 2

Let ALL_{NFA} be the language

$$\{\langle M\rangle \mid M \text{ is an NFA and } L(M) = \Sigma^* \}$$

Theorem: ALL_{NFA} is in co-NSPACE(n). I.e., the complement of ALL_{NFA} is in NSPACE(n).

Note that we still do not know if ALL_{NFA} is in NP, or in co-NP.

Example 2 (cont.)

Proof Idea: We shall construct an NTM S decides the complement of ALL_{NFA}.

To do so, on each input $\langle M \rangle$, we try to find a string that $\langle M \rangle$ rejects so as to show that it is in the complement of ALL_{NFA}.

The NTM S' in the next slide decides the complement of ALL_{NFA}:

Example 2 (cont.)

S' = "On input $\langle M \rangle$,

1. Place a marker on start state of NFA
2. Guess an input string w of length 2^q where q = number of states in M
3. Simulate the running of $\langle M \rangle$ on w, by updating the set of states with marker after reading a character from w
4. If at some point no accept states of M is marked, accept. Otherwise, reject."

Example 2 (cont.)

Question 1: Why is the previous decider correctly decides the complement of ALL_{NFA} ? Note that currently, only strings of length 2^q is examined…

Question 2: Is the space complexity O(length of input)?

Example 2 (cont.)

The previous NTM S′ has space problem…

We now modify it a bit to give S in the next slide, which decides the complement of ALL_{NFA} in O(length of input) space:

Example 2 (cont.)

S' = "On input $\langle M \rangle$,

1. Place a marker on start state of NFA
2. Repeat 2^q times, where q = number of states in M

   (a) Guess the next input symbol and update the set of states with marker to simulate the reading of that symbol
3. If at some point no accept states of M is marked, accept. Otherwise, reject."

Savitch's Theorem

Theorem: Let $f: N \rightarrow R$ be a function, with $f(n) \geq n$. Then,

$$NSPACE(f(n)) \subseteq SPACE((f(n))^2)$$

Proof: Let's do that later ^_^

PSPACE and NSPACE

Definition: PSPACE is the class of languages that are decidable in polynomial space by a DTM. In other words,

$$\text{PSPACE} = \bigcup_k \text{SPACE}(n^k)$$

Similarly, we can define NPSPACE to be the class of languages that are decidable in polynomial space by a NTM. So, what is the relationship between PSPACE and NPSPACE?

PSPACE = NPSPACE

Theorem: PSPACE = NPSPACE

Proof: By Savitch's Theorem.

PSPACE = co-NPSPACE

Theorem: PSPACE = co-NPSPACE

To prove PSPACE $\subseteq$ co-NPSPACE, we see that PSPACE = co-PSPACE (why?), and co-PSPACE $\subseteq$ co-NPSPACE (why?).

To prove co-NPSPACE $\subseteq$ PSPACE, we see that co-NPSPACE $\subseteq$ co-PSPACE (Savitch's Theorem) and PSPACE = co-PSPACE.

P, NP, and PSPACE

Theorem: P $\subseteq$ PSPACE

Proof: If a language is decided by some DTM M in $f(n)$ time, M cannot see more than $f(n)$ cells. Thus, TIME($f(n)$) $\subseteq$ SPACE($f(n)$), so that P $\subseteq$ PSPACE

Theorem: NP $\subseteq$ PSPACE

PSPACE and EXPTIME

Theorem: PSPACE $\subseteq$ EXPTIME

Proof: If a language is decided by some DTM M in $f(n)$ space (where $f(n) \geq n$), M can visit at most $f(n)\, 2^{O(f(n))}$ configurations (why?) Thus, M must run in $f(n)\, 2^{O(f(n))}$ time.

In other words, $\mathrm{SPACE}(f(n)) \subseteq \mathrm{TIME}(2^{O(f(n))})$, so that PSPACE $\subseteq$ EXPTIME.

Summary

$P \subseteq NP \subseteq PSPACE = NPSPACE \subseteq EXPTIME$

It is shown in Chapter 9 that $P \neq EXPTIME$, so that we know at least one of the above containment ($\subseteq$) must be proper ($\subset$)

Unfortunately, at this moment, we still don't know which one(s) is proper. What most researchers believe is all are proper.

Next Time

- Savitch's Theorem
- PSPACE-complete
- L and NL
- NL-complete