CS5371
Theory of Computation

Lecture 21: Complexity VI
(More NP-complete Problems)

Objectives

- Proving a language is NP-complete by reduction
- Examples NP-complete language we shall see include:

  3SAT, CLIQUE, IND-SET,

  VERTEX-COVER, Directed-HAMPATH, HAMPATH, SUBSET-SUM, PARTITION

Conjuctive Normal Form

- A literal is a Boolean variable or a negated Boolean variable. E.g., x, $\neg y$
- A clause is several literals connected with $\vee$'s. E.g., $(x \vee y \vee \neg z)$
- A Boolean formula is in Conjuctive Normal Form (Don't confuse this with Chomosky Normal Form!!!) if it is made of clauses connected with $\wedge$'s.

  E.g., $(x \vee y \vee \neg z) \wedge (\neg y \vee z) \wedge (\neg x)$

CNF-SAT is NP-complete

A Boolean formula is a cnf-formula if it is a formula in Conjuctive Normal Form

Let CNF-SAT be the language

$\{ \langle F \rangle \mid F$ is a satisfiable cnf-formula $\}$

Theorem: CNF-SAT is NP-complete.

CNF-SAT is NP-complete (2)

Proof: To show CNF-SAT is NP-complete, we notice that:

- CNF-SAT is in NP (easy to prove)
- Every language in NP is polynomial time reducible to CNF-SAT → Because the proof of Cook-Levin theorem in Lecture 20 can be directly re-used (recall that the reduction is based on cnf-formula)

Thus, CNF-SAT is NP-complete

3SAT is NP-complete

A Boolean formula is a 3cnf-formula if it is a formula in Conjuctive Normal Form, and every clause has exactly 3 literals

Let 3SAT be the language

{ ⟨F⟩ | F is a satisfiable 3cnf-formula }

Theorem: 3SAT is NP-complete.

3SAT is NP-complete (2)

Proof: To show 3SAT is NP-complete, two things to be done:

- Show 3SAT is in NP (easy)
- Show that every language in NP is polynomial time reducible to 3SAT (how?)
  → It is sufficient to give a polynomial time reduction from some NP-complete language to 3SAT (why?)

Which NP-complete language shall we use?

3SAT is NP-complete (3)

To reduce CNF-SAT to 3SAT, we convert a cnf-formula F into a 3cnf-formula F', such that F is satisfiable if and only if F' is satisfiable

Firstly, let $C_1, C_2, \ldots, C_k$ be the clauses in F. If F is a 3cnf-formula, we just set F' to be F. Otherwise, the following are the only reasons why F is not a 3cnf-formula:

- Some clauses C_i has less than 3 literals
- Some clauses C_i has more than 3 literals

3SAT is NP-complete (4)

We begin with adding a sub-formula to F'.

Let x, y be new variables not in F. The first set of clauses, $(x \vee x \vee y) \wedge (x \vee x \vee \neg y)$, will be added. This ensures that x must be set to 1 for F' to be satisfiable

Now, let us try to replace each of these clauses into an equivalent set of 3-literal-clauses

3SAT is NP-complete (5)

- For each clause that has one literal, say L_1, we change it into $(L_1 \lor L_1 \lor \neg x)$ and add this clause (by AND) to F'. Thus, if F' is satisfiable, the value of L_1 must be 1
- For each clause that has two literals, say $(L_1 \lor L_2)$, we change it into $(L_1 \lor L_2 \lor \neg x)$ and add this clause (by AND) to F'. Thus, if F' is satisfiable, the value of $(L_1 \lor L_2)$ must be 1

3SAT is NP-complete (6)

- For each clause that has more than three literals, say $(L_1 \vee L_2 \vee \ldots \vee L_m)$, we replace it by $(L_1 \vee L_2 \vee z_1) \wedge (\neg z_1 \vee L_3 \vee z_2) \wedge (\neg z_2 \vee L_4 \vee z_3) \wedge \ldots \wedge (\neg z_{m-3} \vee L_{m-1} \vee L_m)$

  and add this set of clauses (by AND) to F'. Thus, if F' is satisfiable, the value of $(L_1 \vee L_2 \vee \ldots \vee L_m)$ must be 1 [why??]

3SAT is NP-complete (7)

- Finally, for each clause that has three literals, we simply add this clause (by AND) to F'. Thus, if F' is satisfiable, the value of this clause must be 1
- By our construction of F', F is satisfiable if and only if F' is satisfiable (why??)
- Also, the above conversion takes polynomial time (why??). Thus, we show a polynomial time reduction from CNF-SAT to 3SAT → 3SAT is NP-complete

CLIQUE is NP-complete

Recall that CLIQUE is the language

$\{ \langle G,k \rangle \mid G$ is a graph with a k-clique $\}$

Theorem: CLIQUE is NP-complete.

How to prove??

CLIQUE is NP-complete (2)

Proof: To show CLIQUE is NP-complete, two things to be done:

- Show CLIQUE is in NP (done before)
- Show that every language in NP is polynomial time reducible to CLIQUE
  ➔ It is sufficient to give a polynomial time reduction from some NP-complete language to CLIQUE

Which NP-complete language shall we use?

CLIQUE is NP-complete (3)

Let us try to reduce 3SAT to CLIQUE:

Let F be a 3cnf-formula. Let $C_1, C_2, \ldots, C_k$ be the clauses in F.

Hint: Construct a graph G such that F is satisfiable if and only if G has a k-clique

CLIQUE is NP-complete (4)

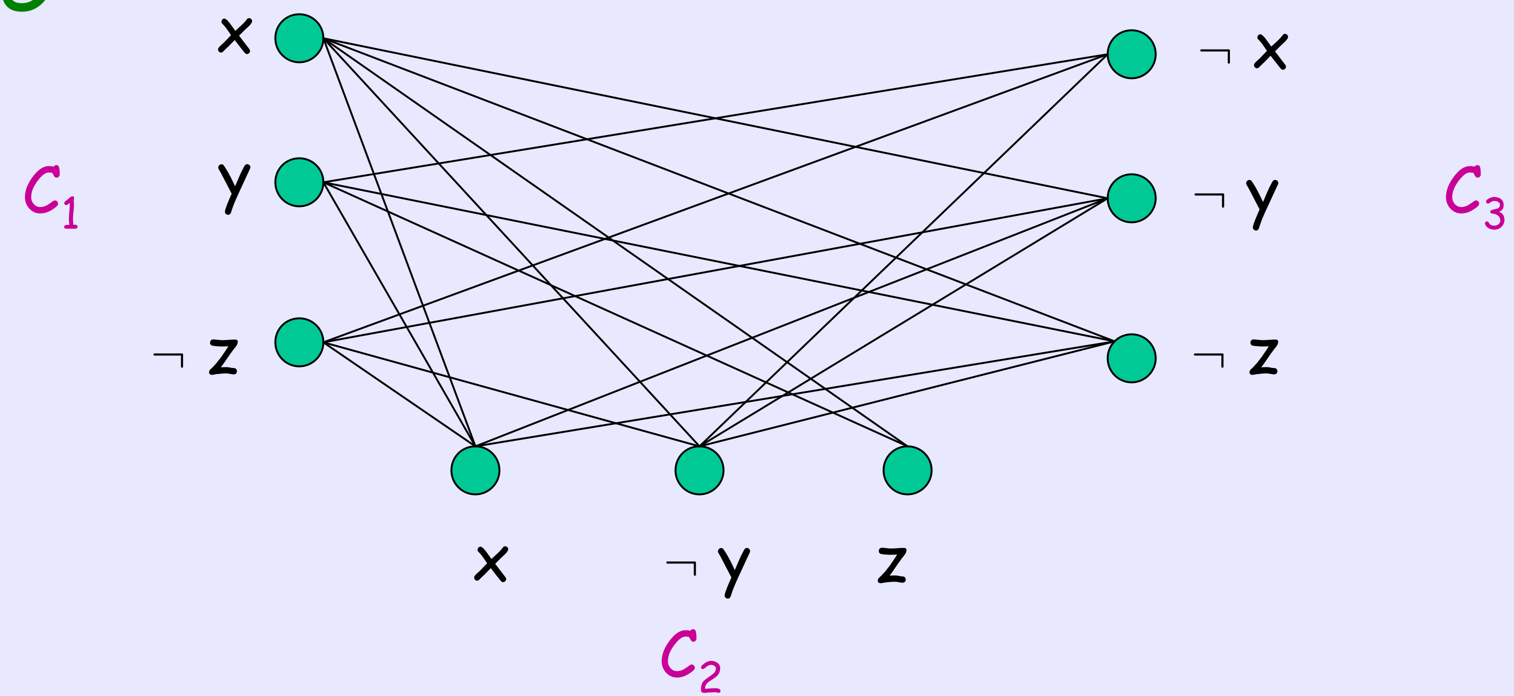
Proof (cont.): Let F be a 3cnf-formula. Let $C_1, C_2, \ldots, C_k$ be the clauses in F. For each clause C_j, let $x_{j,1}$, $x_{j,2}$, $x_{j,3}$ be its literals.

We construct a graph as follows: For each literal $x_{j,q}$, we create a distinct vertex in G representing it. G contains all possible edges except those joining two vertices in the same clause, and except those joining two vertices whose literals is the negation of the others. E.g., (next slide)

Constructing G from F

$$F = (x \vee y \vee \neg z) \wedge (x \vee \neg y \vee z) \wedge (\neg x \vee \neg y \vee \neg z)$$

G



CLIQUE is NP-complete (5)

Proof (cont.): We now show that G has a k-clique if and only if F is satisfiable.

➔ If G has a k-clique, the k-clique must a vertex from each clause (why?) Also, no vertex will be the negation of the others in the clique (why?) Thus, by setting the corresponding literal (not variable) to TRUE, each clause in F will be satisfied.

CLIQUE is NP-complete (6)

← If F is satisfiable, (at least) a literal in each clause will be set to TRUE in the satisfying assignment. The corresponding vertices in G must form a clique (why?) Thus, G has a k-clique.

Notice that G can be constructed from F in polynomial time → We have a polynomial time reduction from 3SAT to CLIQUE → Thus, CLIQUE is NP-complete

IND-SET is NP-complete

A set of vertices inside a graph G is an independent set if there are no edges between any two of these vertices.

Let IND-SET be the language

$\{ \langle G,k \rangle \mid G$ is a graph with an independent set of size $k \}$

Theorem: IND-SET is NP-complete.

IND-SET is NP-complete (2)

Proof: To show IND-SET is NP-complete, two things to be done:

- Show IND-SET is in NP (easy)
- Show every language in NP is polynomial time reducible to IND-SET
  ➔ It is sufficient to give a polynomial time reduction from some NP-complete language to IND-SET

Hint: Use CLIQUE for the reduction

IND-SET is NP-complete (3)

Proof (cont.): We now show that a problem in CLIQUE can be reduced to a problem in IND-SET in polynomial time.

We shall construct G' such that G has a k-clique if and only if G' has an independent set of size k. That is, construct G' such that

$$\langle G,k \rangle \text{ in CLIQUE} \Leftrightarrow \langle G',k \rangle \text{ in IND-SET}$$

IND-SET is NP-complete (4)

Given $G=(V,E)$, we set $G'=(V',E')$ to be the complement of G. In other words, $V = V'$ (G and G' has the same set of vertices), but e in $E \Leftrightarrow e$ not in E'

It is easy to check that G' is the desired graph we want (how to check?). As the construction of G' is done in polynomial time, we have a polynomial time reduction from CLIQUE to IND-SET → IND-SET is NP-complete.

VERTEX-COVER is NP-complete

A set of vertices inside a graph G is a vertex cover if every edge in G is connected to at least one vertex in the set.

Let VERTEX-COVER be the language

$\{ \langle G,k \rangle \mid G$ is a graph with a vertex cover of size $k \}$

Theorem: VERTEX-COVER is NP-complete.

VERTEX-COVER is NP-complete (2)

Proof: To show VERTEX-COVER is NP-complete, two things to be done:

- Show VERTEX-COVER is in NP (easy)
- Show that every language in NP is polynomial time reducible to VERTEX-COVER → It is sufficient to give a polynomial time reduction from some NP-complete language to VERTEX-COVER

Hint: Use IND-SET for the reduction

VERTEX-COVER is NP-complete (3)

Proof (cont.): We now show that a problem in IND-SET can be reduced to a problem in VERTEX-COVER in polynomial time.

In fact, for G having n vertices, we will simply show that G has an independent set of size k if and only if G has a vertex cover of size n-k. That is, we show

$$\langle G,k\rangle \text{ in IND-SET} \Leftrightarrow \langle G,n\text{-}k\rangle \text{ in VERTEX-COVER}$$

VERTEX-COVER is NP-complete (4)

Given $G=(V,E)$, if V' is a vertex cover, then every edge is attached to at least one vertex in V'. By deleting V' from the graph, no edge remains. Thus, $V-V'$ will be an independent set. On the other hand, if $V-V'$ is an independent set, V' must be a vertex cover (why?).

Thus, we have a polynomial time reduction from IND-SET to VERTEX-COVER → VERTEX-COVER is NP-complete.

Next Time

- More NP-complete problems