CS2351
Data Structures

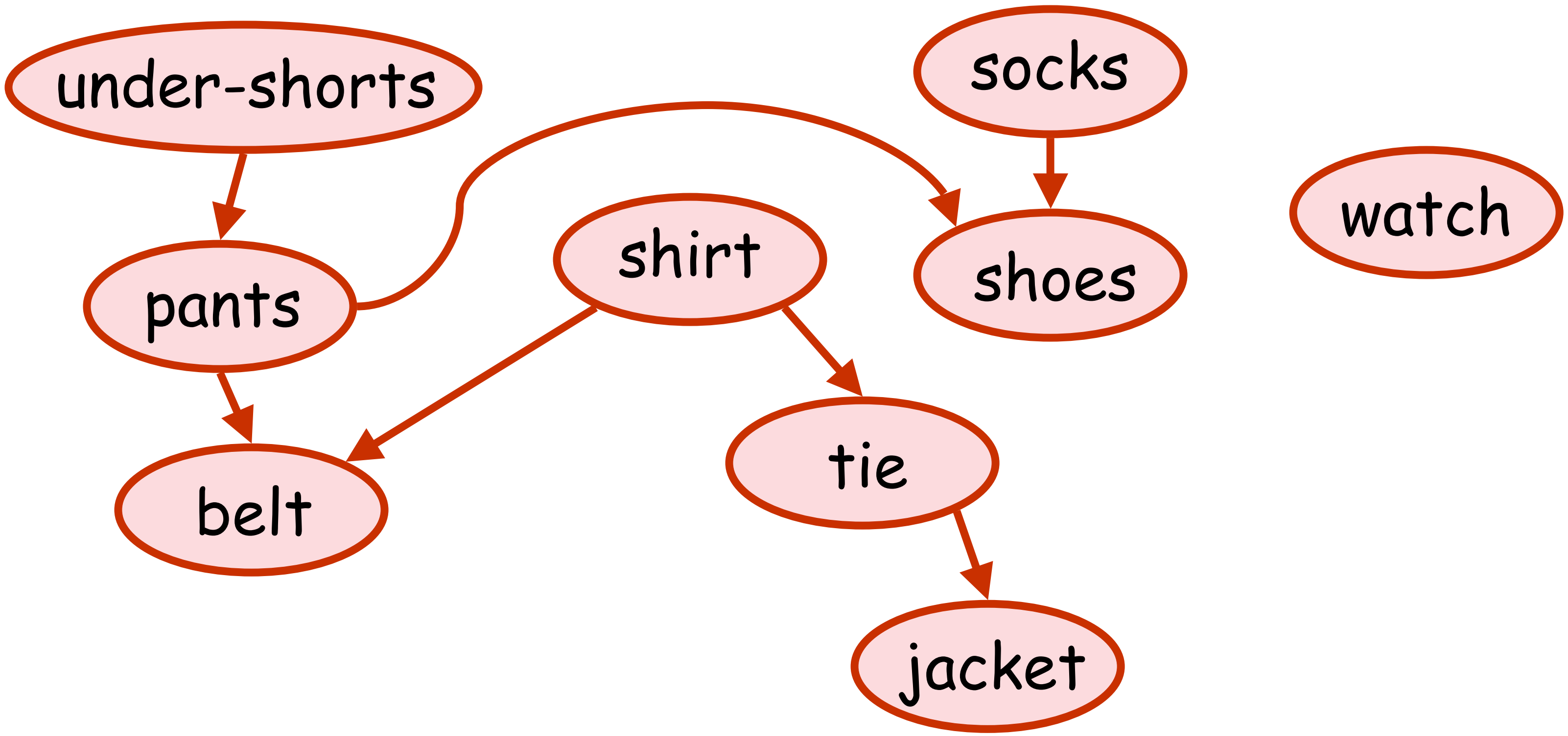
Lecture 12:
Graph and Tree Traversals III

About this lecture

- We introduce the Topological Sort problem on directed acyclic graph (DAG)
- We give two linear-time algorithms :
  (1) Using Queue
  (2) Using Stack

Topological Sort

- Directed graph can be used to indicate precedence among a set of events
- E.g., a possible precedence is dressing



Topological Sort

- The previous directed graph is also called a precedence graph

Question: Given a precedence graph G, can we order the events such that if (u,v) is in G (i.e. u should complete before v) then u appears before v in the ordering ?

We call this problem topological sorting of G

Topological Sort

Fact: If G contains a cycle, then it is impossible to find a desired ordering

(Prove by contradiction)

- However, if G is acyclic (contains no cycles) we shall give two algorithms that always find the desired ordering

Topological Sort (with Queue)

```
Topological-Sort(G)   // given G is acyclic
{
   while (G contains a vertex)
   {  1. Pick a vertex v with in-degree = 0 ;
      2. Remove all its outgoing edges ;
      3. Output v ;
   }
}
```

Why is the algorithm correct?

Topological Sort (with Queue)

Theorem:

If G is acyclic, the previous algorithm produces a topological sort of G

Proof:

Two cases may happen when we run the previous algorithm.

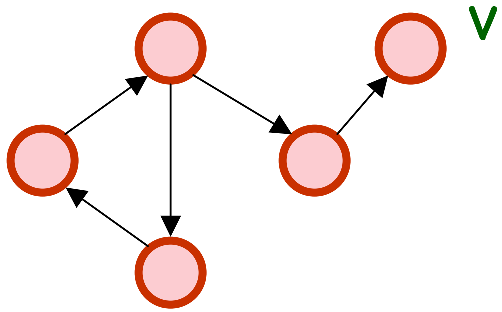
Case 1 : All vertices are output

Case 2 : Some vertex may not be output

Proof

- In Case 1, vertices are sorted correctly

- In Case 2, the remaining vertices must each have in-degree $\geq$ 1. Now, we pick a vertex v in this group, repeatedly visit another vertex by tracing an incoming edge, some vertex will be visited twice (why?) ➔ a cycle is found !!



Performance

- Let $G = (V,E)$ be the input directed graph
- Running time for Topological-Sort :

 1. Each vertex keeps # incoming edges
 2. Finding vertices with in-degree = 0 :

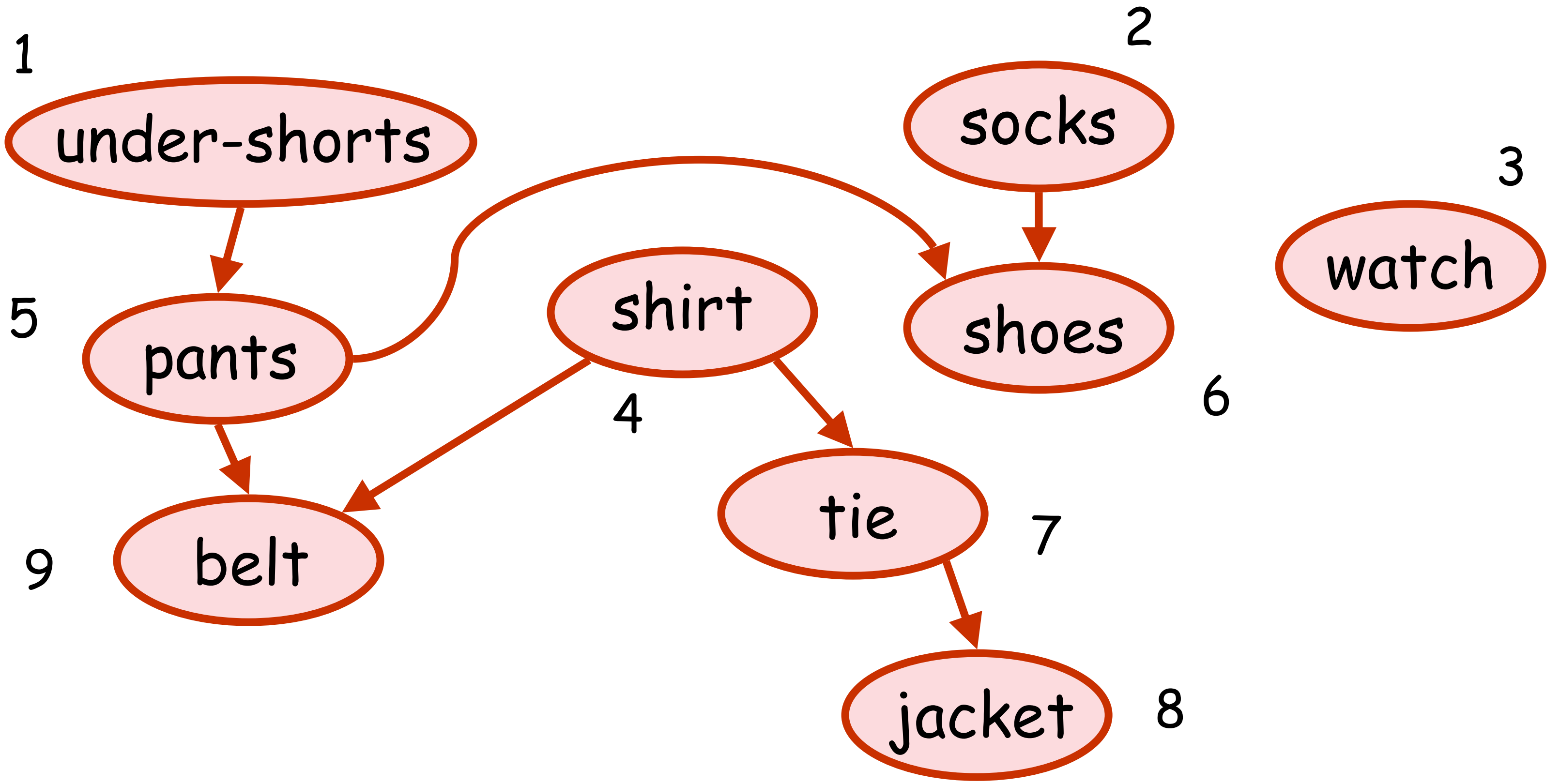
    Naïve method: $O(|V|^2)$ total time

    Clever method: (use a queue Q)

    Enqueue vertex once its in-degree = 0

- Total time: $O(|V|+|E|)$

Topological Sort (Example)



When a vertex is output,
its indegree is 0

Topological Sort (with Stack)

Topological-Sort(*G*) `// given G is acyclic`

{

1. Call DFS on *G*
2. Output vertices in decreasing order of their finishing times ;

}

Why is the algorithm correct?

Topological Sort (with Stack)

Theorem:

If G is acyclic, the previous algorithm produces a topological sort of G

Proof: Let (u,v) be an edge. We shall show $f(v) < f(u)$ so that the ordering is correct.

Firstly, during DFS, there are two cases

- Case 1 : u is visited before v
- Case 2 : v is visited before u

Proof

- In Case 1, u cannot finish before DFS is performed on all its neighbors. Since v is a neighbor of u , we must have

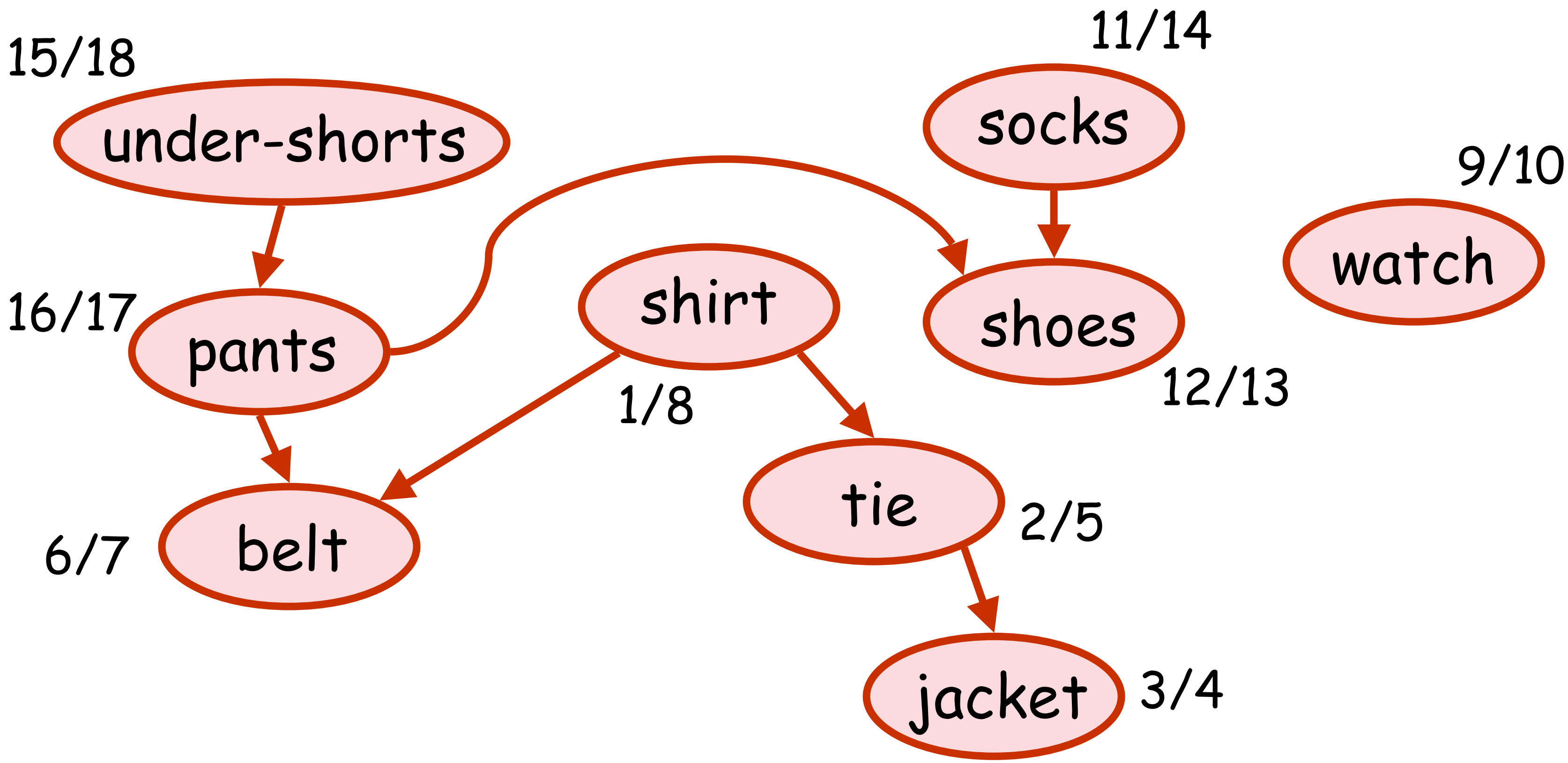
$$d(u) < d(v) < f(v) < f(u)$$

- In Case 2, v must have finished before u starts (else, there will be a path from v to u and the graph contains a cycle.) Thus,

$$f(v) < d(u) \Rightarrow f(v) < f(u)$$

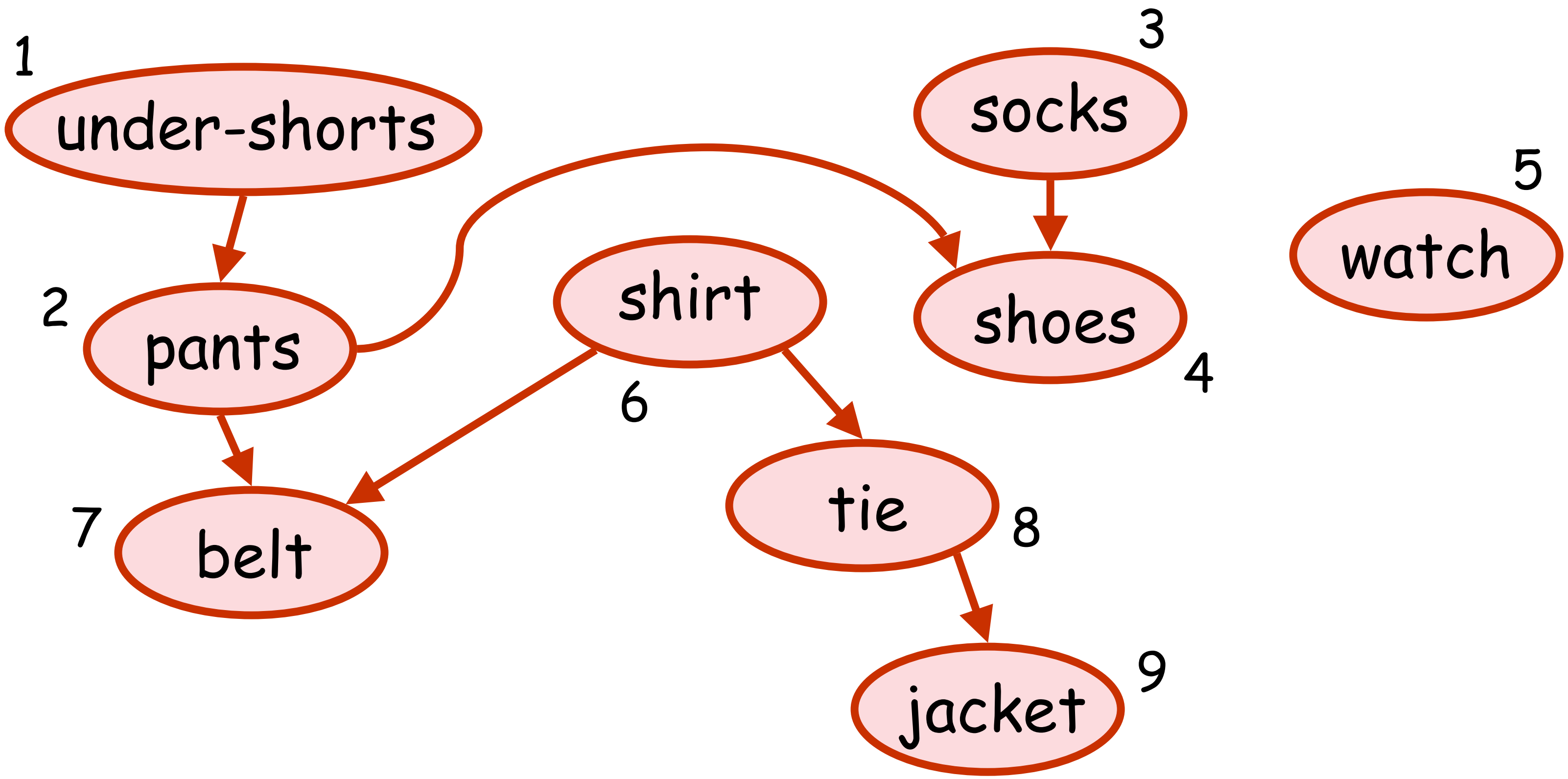
- Both cases show $f(v) < f(u) \Rightarrow$ Done !

Topological Sort (Example)



Discovery and Finishing Times
after a possible DFS

Ordering Finishing Times (in descending order)



If we order the events from left to right, anything special about the edge directions ?

Performance

- Let $G = (V,E)$ be the input directed graph
- Running time for Topological-Sort :

1. Perform DFS : $O(|V|+|E|)$ time
2. Sort finishing times

   Naïve method: $O(|V| \log |V|)$ time

   Clever method: (use an extra stack S)

   During DFS, push a node into stack S once finished → no need to sort !!

- Total time: $O(|V|+|E|)$