CS2351
Data Structures

Tutorial 3:
Data Structures for Disjoint Sets

About this lecture

- Data Structure for Disjoint Sets
  - Support Union and Find operations
- Various Methods:
  1. Union by Size
  2. Union by Rank
  3. Union by Rank + Path Compression

Maintaining Disjoint Set

- In some applications, especially in algorithms relating to graphs, we often have a set of elements, and want to maintain a dynamic partition of them
  - I.e., the partition changes over time
- Our target corresponds to maintaining dynamic disjoint sets of the elements

Maintaining Disjoint Set

- Let $\Sigma = \{ S_1, S_2, \ldots, S_k \}$ be a collection of dynamic disjoint sets of the elements
- Let x and y be any two elements
- We want to support:

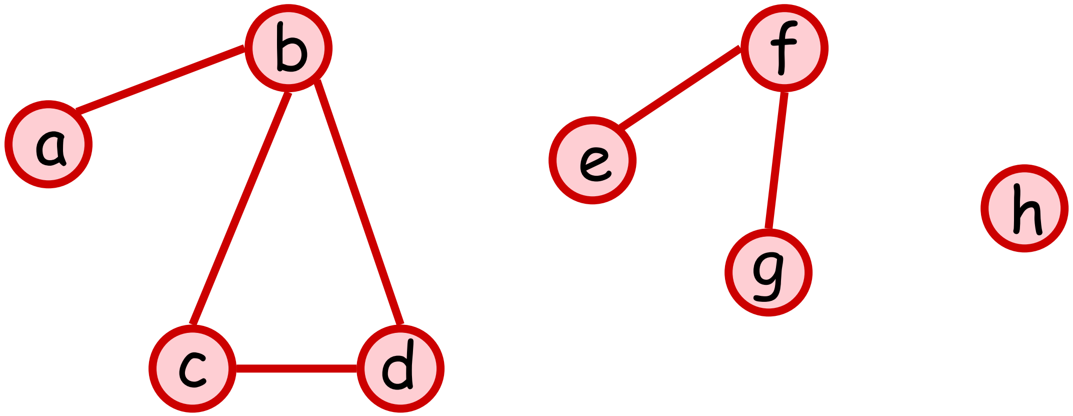
Make-Set(x): create a set containing x

Find(x) : return which set x belongs

Union(x,y) : merge the sets containing x and containing y into one

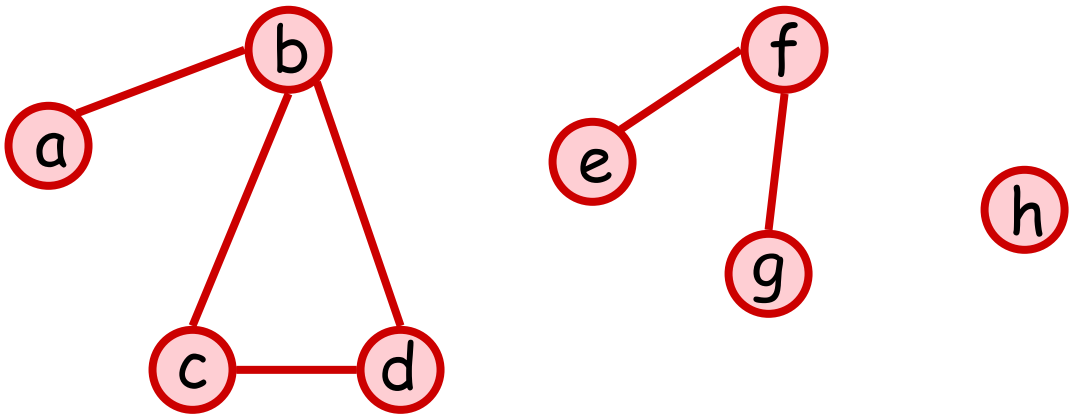
Example Application: Finding Connected Components

Step 0: Begin with the input graph



Example Application: Finding Connected Components

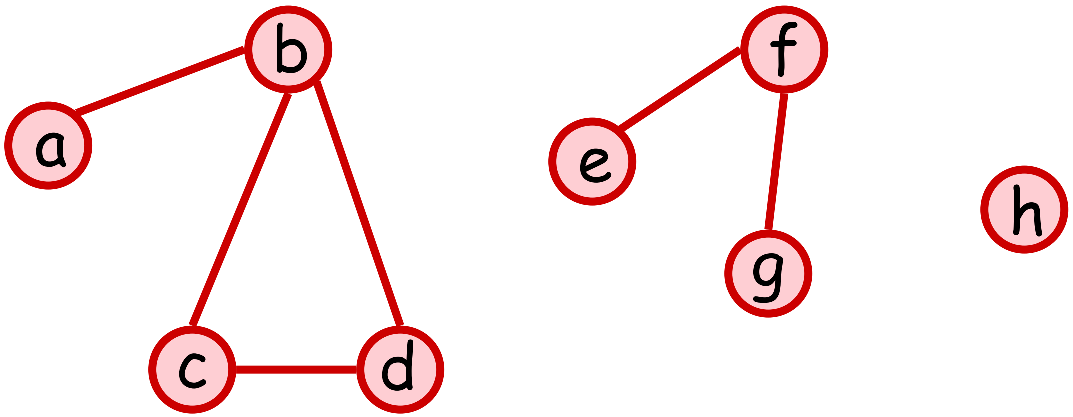
Step 1: Make-Set(v) for each vertex v



current Σ: { {a}, {b}, {c}, {d}, {e}, {f}, {g}, {h} }

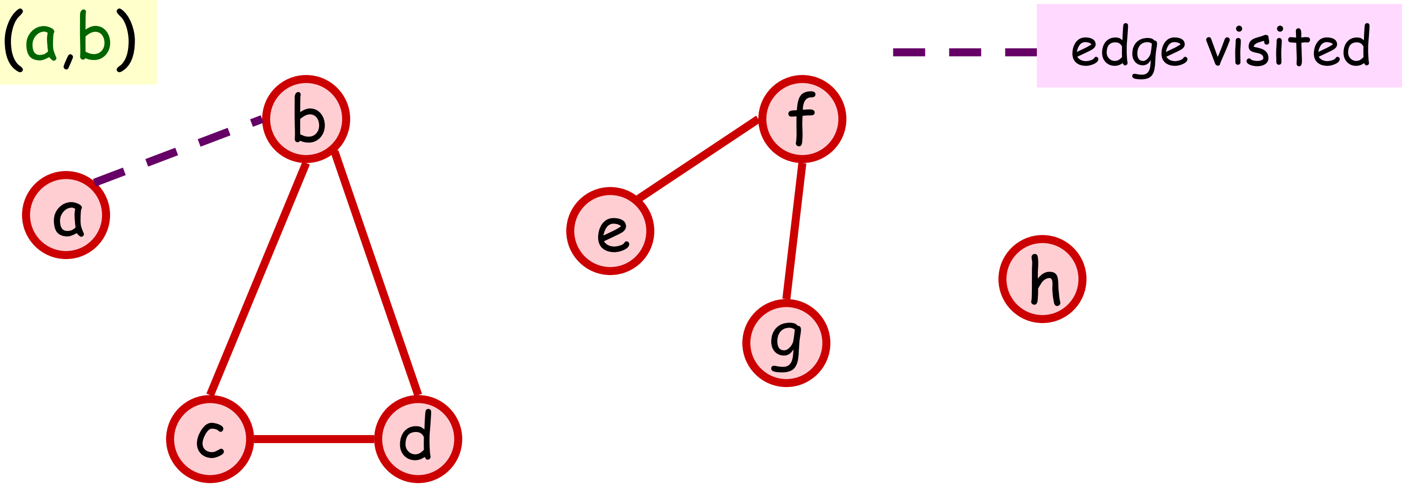
Example Application: Finding Connected Components

Step 2: Visit each edge (u,v), perform Union(u,v)



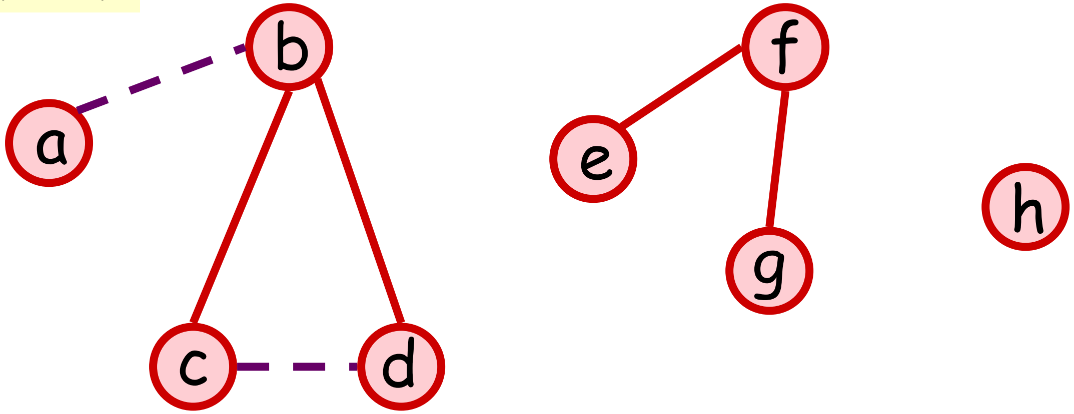
current Σ: { {a}, {b}, {c}, {d}, {e}, {f}, {g}, {h} }

Step 2: Visit (a,b)



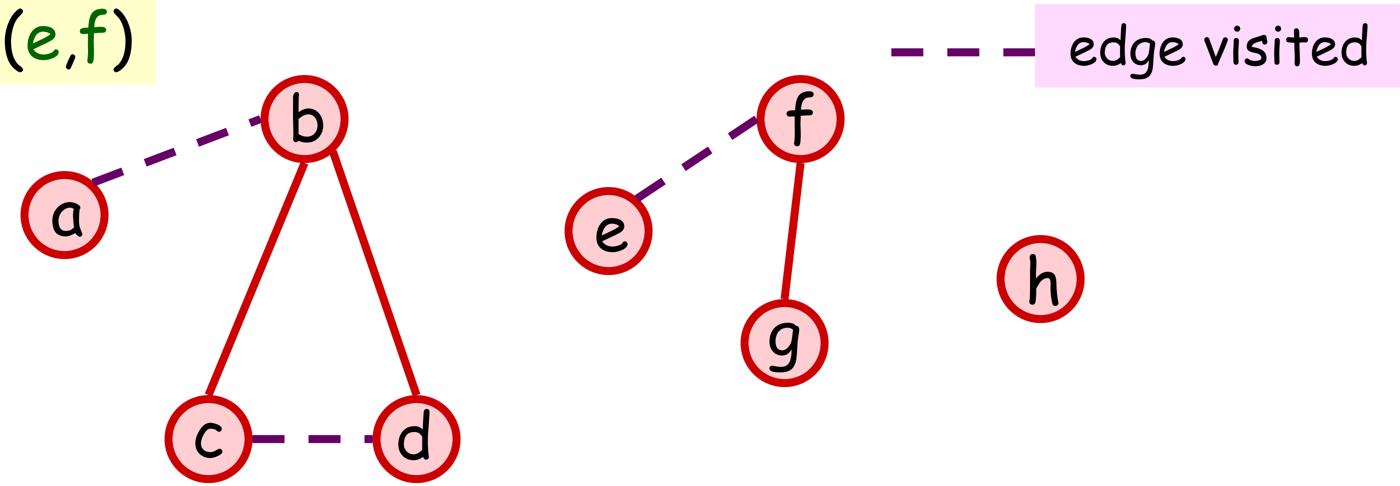
current Σ: { {a,b}, {c}, {d}, {e}, {f}, {g}, {h} }

Step 2: Visit (c,d)



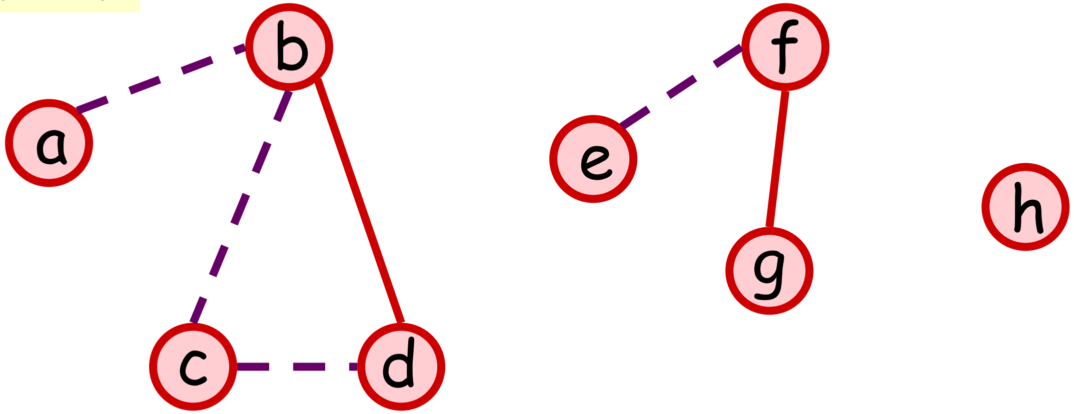
current Σ: { {a,b}, {c,d}, {e}, {f}, {g}, {h} }

Step 2: Visit (e,f)



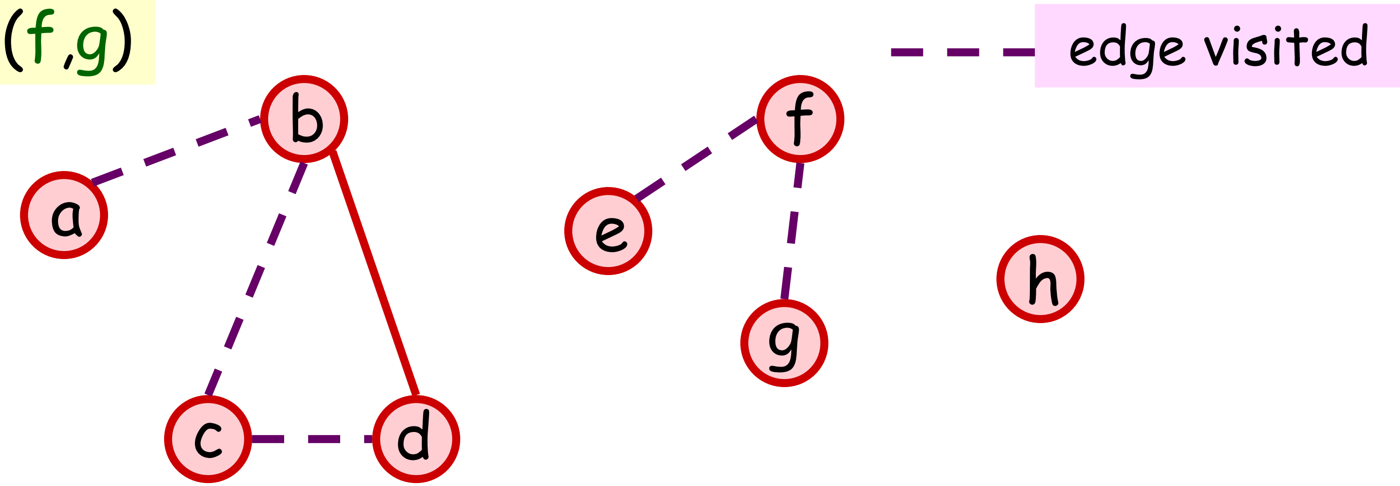
current Σ: { {a,b}, {c,d}, {e,f}, {g}, {h} }

Step 2: Visit (b,c)



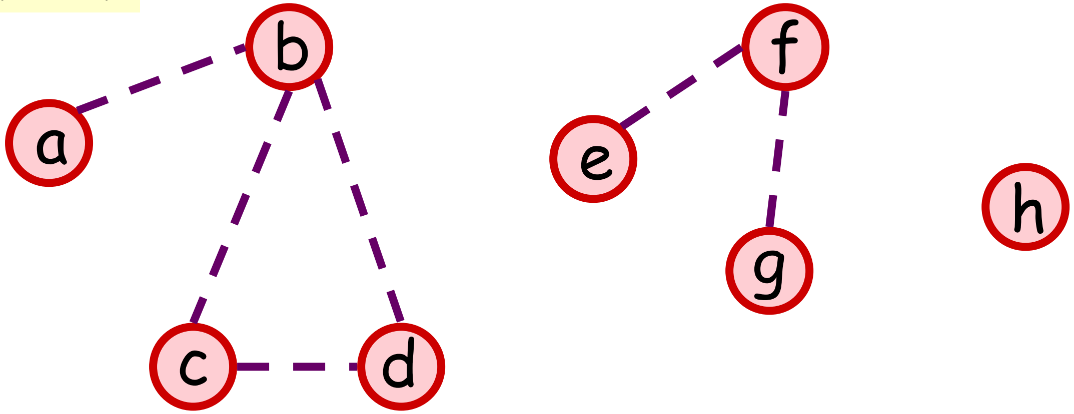
current Σ: { {a,b,c,d}, {e,f}, {g}, {h} }

Step 2: Visit (f,g)



current Σ: { {a,b,c,d}, {e,f,g}, {h} }

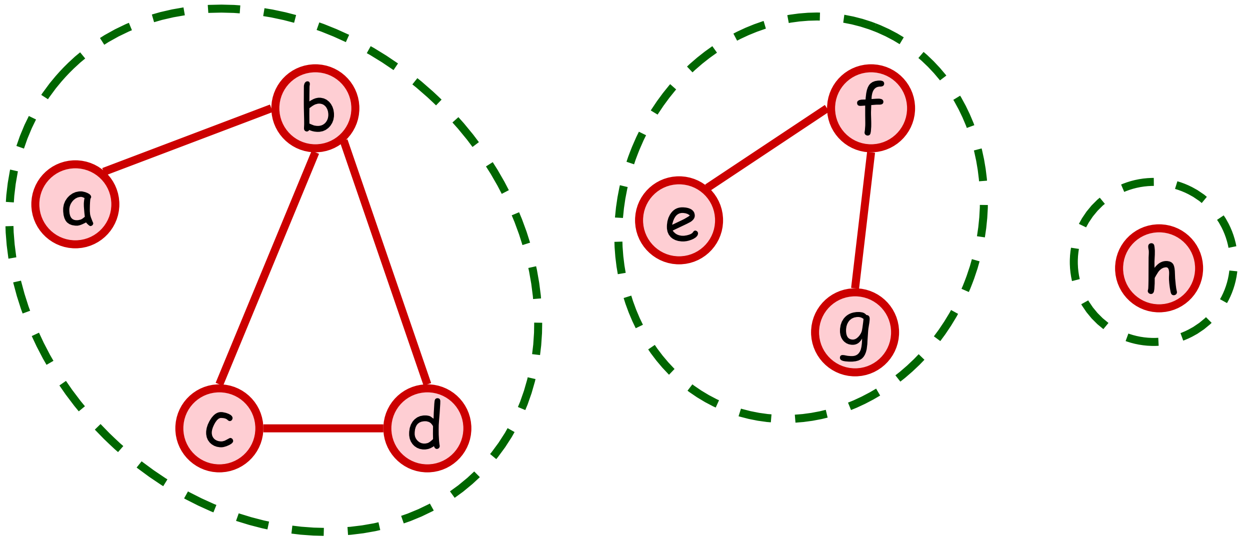
Step 2: Visit (b,d)



current Σ: { {a,b,c,d}, {e,f,g}, {h} }

Example Application: Finding Connected Components

After Step 2 (when all edges visited) :
Each Disjoint Set ⇔ Connected Component



current Σ: { {a,b,c,d}, {e,f,g}, {h} }

Remarks

- To facilitate Find(x), each set usually chooses one of its element as a representative

  ➔ Find(x) returns the representative element of the set where x belongs

- To check if x and y belong to the same set, we can just check if

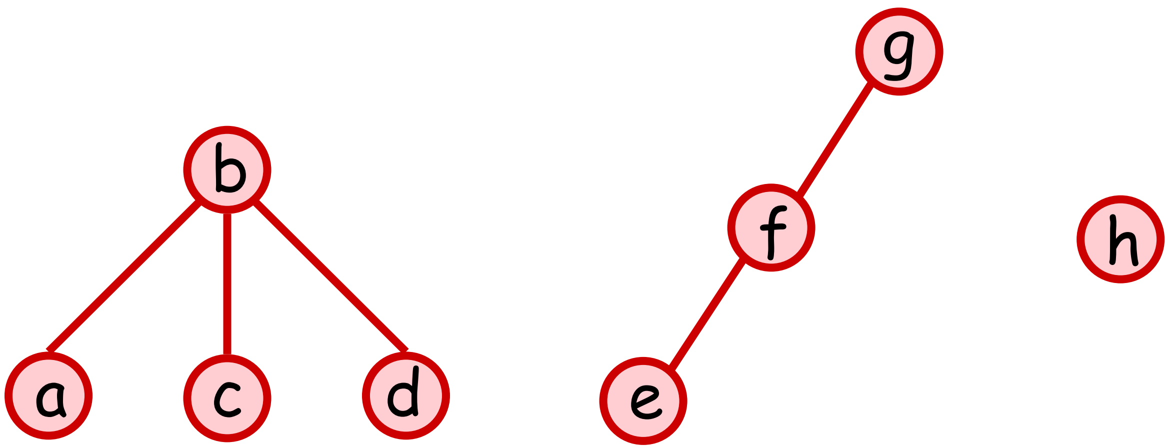
  Find(x) == Find(y)

Disjoint-Set Forest

- One popular method to maintain disjoint sets is by a forest
  - Each set ⇔ a separate rooted tree
  - Representative ⇔ root of tree

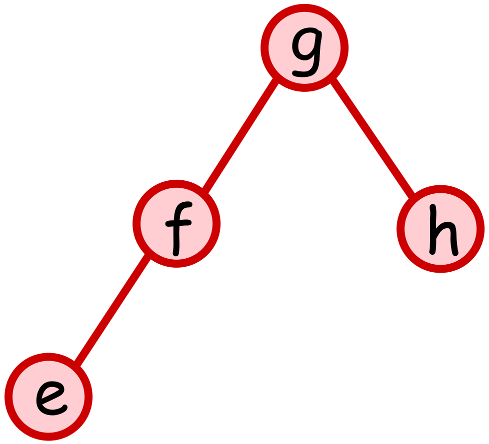
Example

Current dynamic sets : { {a,b,c,d}, {e,f,g}, {h} }



Disjoint-Set Forest

- To perform Union(x,y), we join the trees containing x and containing y, by linking their roots
- E.g. Union(f,h) in previous example gives:



Disjoint-Set Forest

- Let H_{max} = max height of all trees
- In the worst-case:

Make-Set : $\Theta(1)$ time

Find or Union : $O(H_{max})$ time

➔ m operations on n elements :

worst-case $\Theta(mn)$ time

Union By Size

- Let us apply a union-by-size heuristic :

To perform Union, we link root of the smaller tree to root of the larger tree

➔ $H_{max} = O(\log n)$ (how to prove??)

➔ m operations : $\Theta(m \log n)$ time

Union By Rank

- A similar heuristic is called union-by-rank
- Each node keeps track of its rank – an upper bound on the height of the node
  - In a single-node tree (created by Make-Set)

rank of root = 0

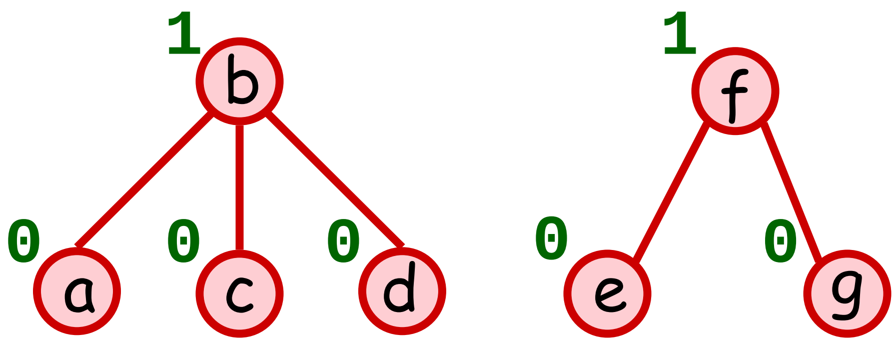
To perform Union, we link root with smaller rank to root with larger rank

Union By Rank

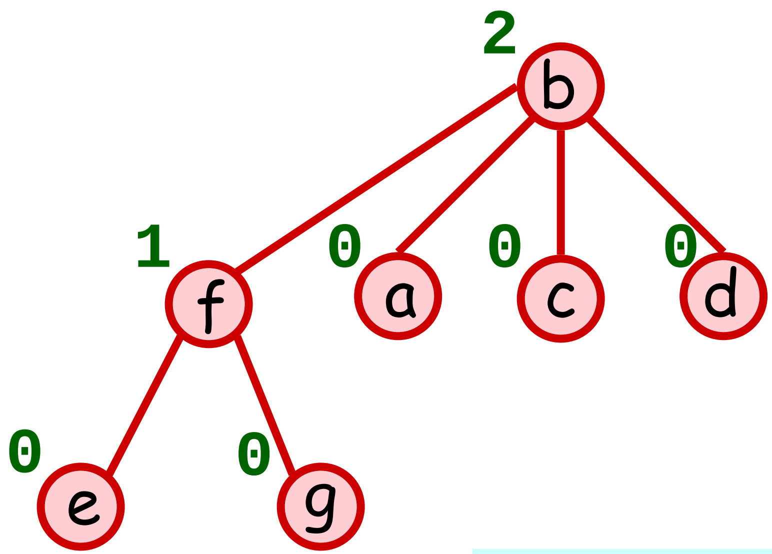
- Rank needs not be very accurate
  - as long as it always gives an upper bound of height is enough
- When Union is performed, only the rank of the roots may change :
  - If both roots have same rank
    ➔ rank of new root increases by 1
  - Else, no change

Example of Union by Rank

Before Union



After Union(c,f)



? = rank

Union By Rank

- Let H_{max} = max height of all trees

  ➔ $H_{max} = O(\log n)$ (how to prove??)

  ➔ m operations : $\Theta(m \log n)$ time

- So, union by rank is no better than union by size, but …

Path Compression

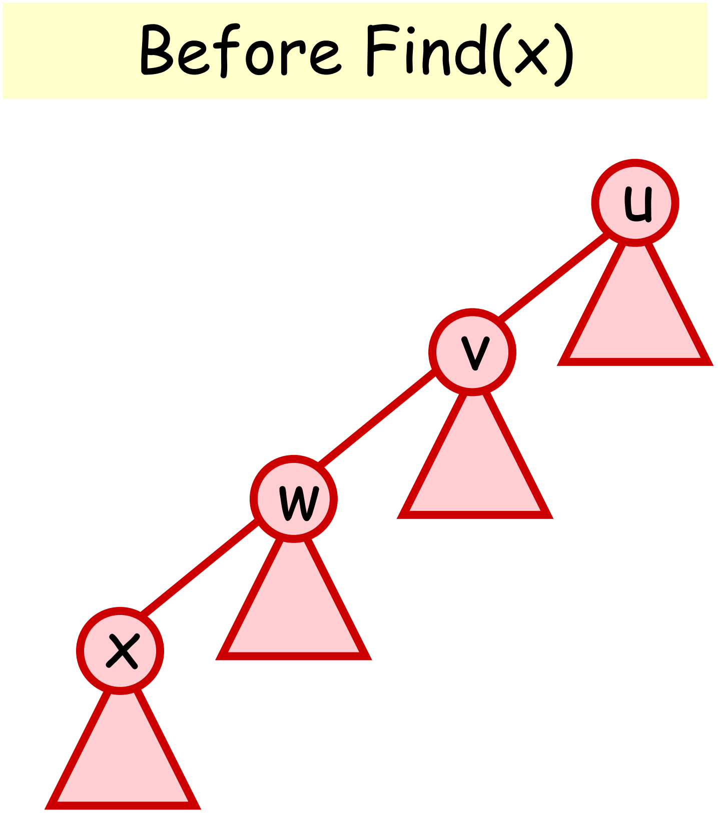
- The closer a node to its root, the faster the Find or Union operation
- When we perform Find(x), we will need to find the root of the tree containing x

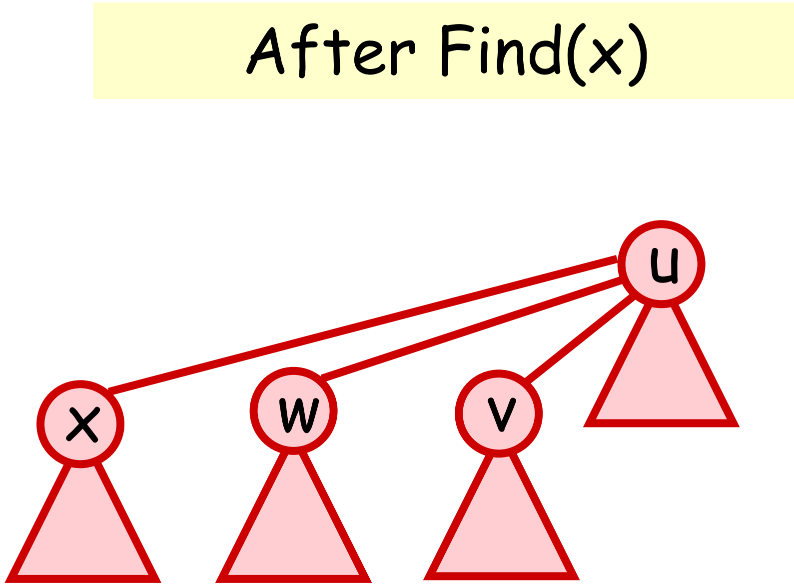
  ➔ will access every ancestor of x

  - why don't we make all these ancestors of x closer to the root now?

    (Because no increase in asymptotic performance !!!)

Example of Path Compression





Union by Rank + Path Compression

- If Union(x,y) is always performed by first Find(x), Find(y), and then linking the roots, then by combining union-by-rank (at Union) and path compression (at Find and Union) :

m operations: $\Theta(m\,\alpha(n))$ time

Inverse Ackermann
(in practice, at most 4)

Finding Connected Components

- Recall: To find connected components of a graph G with n vertices and m edges
  - there are n Make-Set and m Find or Union operations

- Which scheme for dynamic disjoint sets gives the best running time (theoretically)?

Ans. Depends on m (why?)