CS2351
Data Structures

Lecture 16:
Probability & Expectation

Our Status on Searching

- So far, we have studied :
  - BST (such as AVL Tree or RB Tree) which has O(log n) search time, and
  - B-Tree which has $O(\log_B n)$ search I/Os
- They are optimal in the comparison model

- Questions :
  - Can we search faster in other models ?
  - What if only part of the operations are needed (e.g., don't need predecessor) ?

Our Status on Searching

- In the coming lectures, we shall study an interesting searching topic called hashing
  - Target : to store a set of data so that search and updates are done in expected O(1) time
  - In a special case, we show that worst case O(1) search time is possible !
- Before that, let us briefly review about probability and expectation

About this lecture

- What is Probability ?
- What is an Event ?
- What is a Random Variable (RV) ?
- What is Expectation of a RV ?
- Useful Theorem:

  Linearity of Expectation

Experiment and Sample Space

- An experiment is a process that produces an outcome
- A random experiment is an experiment whose outcome is not known until it is observed
  - Exp 1: Throw a die once
  - Exp 2: Flip a coin until Head comes up

Experiment and Sample Space

- A sample space Ω of a random experiment is the set of all outcomes
  - Exp 1: Throw a die once
  - Sample space: { 1, 2, 3, 4, 5, 6 }
  - Exp 2: Flip a coin until Head comes up
  - Sample space: ??
- Any subset of sample space Ω is called an event

Probability

- Probability studies the chance of each event occurring
- Informally, it is defined with a function Pr that satisfies the following:

(1) For any event E, $0 \leq Pr(E) \leq 1$

(2) $Pr(\Omega) = 1$

(3) If E_1 and E_2 do not have common outcomes,

$$Pr(E_1 \cup E_2) = Pr(E_1) + Pr(E_2)$$

Example

Questions:

1. Suppose the die is a fair die, so that
   Pr(1)= Pr(2) = … = Pr(6).
   What is Pr(1)? Why?

2. Instead, if we know
   Pr(1) = 0.2, Pr(2) = 0.3, Pr(3) = 0.4,
   Pr(4) = 0.1, Pr(5) = Pr(6) = 0.
   What is Pr({1,2,4})?

Random Variable

Definition: *A* random variable X on a sample space Ω is a function that maps each outcome of Ω into a real number. That is, X: Ω → $\mathcal{R}$.

Ex: Suppose that we throw two dice

➔ Ω = { (1,1), (1,2), …, (6,5), (6,6) }

Define X = sum of outcome of two dice

➔ X is a random variable on Ω

Random Variable

- For a random variable X and a value a, the notation

  "X = a"

  denotes the set of outcomes ω in the sample space such that X(ω) = a

  ➔ "X = a" is an event

- In previous example,

  "X = 10" is the event {(4,6), (5,5), (6,4)}

Expectation

Definition: The expectation (or average value) of a random variable X, is

$$E[X] = \sum_i i \Pr(X=i)$$

Question:

- X = sum of outcomes of two fair dice
  What is the value of E[X] ?
- How about the sum of three dice?

Expectation (Example)

Let X = sum of outcomes of two dice.

The value of X can vary from 2 to 12

So, we calculate:

$Pr(X=2) = 1/36$, $Pr(X=3) = 2/36$,

$Pr(X=4) = 3/36$, ... , $Pr(X=12) = 2/36$,

$$E[X] = 2*Pr(X=2) + 3*Pr(X=3) + \ldots + 11*Pr(X=11) + 12*Pr(X=12) = 7$$

Linearity of Expectation

Theorem: Given random variables X_1, X_2, …, X_k, each with finite expectation, we have

$$E[X_1+X_2+\ldots+X_k] = E[X_1]+E[X_2] +\ldots+ E[X_k]$$

Let X = sum of outcomes of two dice.

Let X_i = the outcome of the i^{th} dice

What is the relationship of X, X_1, and X_2?

Can we compute $E[X]$?

Linearity of Expectation (Example)

Let X = sum of outcomes of two dice.

Let X_i = the outcome of the i^{th} dice

➔ $X = X_1 + X_2$

➔ $E[X] = E[X_1+X_2] = E[X_1] + E[X_2]$

$= 7/2 + 7/2 = 7$

Can you compute the expectation of the sum of outcomes of three dice?