CS2351
Data Structures

Lecture 16:
Probability & Expectation
Our Status on Searching

• So far, we have studied:
  - **BST** (such as AVL Tree or RB Tree) which has $O(\log n)$ search time, and
  - **B-Tree** which has $O(\log_B n)$ search I/Os

• They are optimal in the comparison model

• Questions:
  - Can we search faster in other models?
  - What if only part of the operations are needed (e.g., don’t need predecessor)?
Our Status on Searching

• In the coming lectures, we shall study an interesting searching topic called **hashing**
  - Target: to store a set of data so that search and updates are done in expected $O(1)$ time
  - In a special case, we show that worst case $O(1)$ search time is possible!

• Before that, let us briefly review about probability and expectation
About this lecture

- What is **Probability**?
- What is an **Event**?
- What is a **Random Variable (RV)**?
- What is **Expectation** of a **RV**?
- Useful Theorem:
  
  **Linearity of Expectation**
Experiment and Sample Space

- An experiment is a process that produces an outcome.
- A random experiment is an experiment whose outcome is not known until it is observed.
  - Exp 1: Throw a die once
  - Exp 2: Flip a coin until Head comes up
Experiment and Sample Space

- A sample space $\Omega$ of a random experiment is the set of all outcomes
  - Exp 1: Throw a die once
    - Sample space: $\{ 1, 2, 3, 4, 5, 6 \}$
  - Exp 2: Flip a coin until Head comes up
    - Sample space: ??

- Any subset of sample space $\Omega$ is called an event
Probability

• Probability studies the chance of each event occurring

• Informally, it is defined with a function \( \text{Pr} \) that satisfies the following:

(1) For any event \( E \), \( 0 \leq \text{Pr}(E) \leq 1 \)

(2) \( \text{Pr}(\Omega) = 1 \)

(3) If \( E_1 \) and \( E_2 \) do not have common outcomes,
\[ \text{Pr}(E_1 \cup E_2) = \text{Pr}(E_1) + \text{Pr}(E_2) \]
Example

Questions:

1. Suppose the die is a \textit{fair} die, so that
   \[ \Pr(1) = \Pr(2) = \ldots = \Pr(6). \]
   What is \( \Pr(1) \)? Why?

2. Instead, if we know
   \[ \Pr(1) = 0.2, \Pr(2) = 0.3, \Pr(3) = 0.4, \]
   \[ \Pr(4) = 0.1, \Pr(5) = \Pr(6) = 0. \]
   What is \( \Pr(\{1,2,4\}) \)?
Definition: A random variable \( X \) on a sample space \( \Omega \) is a function that maps each outcome of \( \Omega \) into a real number. That is, \( X: \Omega \rightarrow \mathbb{R} \).

Ex: Suppose that we throw two dice

\[ \Omega = \{ (1,1), (1,2), \ldots, (6,5), (6,6) \} \]

Define \( X = \text{sum of outcome of two dice} \)

\( X \) is a random variable on \( \Omega \)
Random Variable

• For a random variable $X$ and a value $a$, the notation

"$X = a$"

denotes the set of outcomes $\omega$ in the sample space such that $X(\omega) = a$

$\Rightarrow$ "$X = a$" is an event

• In previous example,

"$X = 10$" is the event $\{(4,6), (5,5), (6,4)\}$
Expectation

Definition: The expectation (or average value) of a random variable $X$, is

$$E[X] = \sum_i i \Pr(X=i)$$

Question:

- $X =$ sum of outcomes of two fair dice
  What is the value of $E[X]$?
- How about the sum of three dice?
Expectation (Example)

Let $X = \text{sum of outcomes of two dice.}$

The value of $X$ can vary from 2 to 12.

So, we calculate:

$Pr(X=2) = 1/36$, $Pr(X=3) = 2/36$, 
$Pr(X=4) = 3/36$, ... , $Pr(X=12) = 2/36$,

$E[X] = 2*Pr(X=2) + 3*Pr(X=3) + ... + 11*Pr(X=11) + 12*Pr(X=12)$

$= 7$
Linearity of Expectation

Theorem: Given random variables $X_1, X_2, \ldots, X_k$, each with finite expectation, we have

$$E[X_1 + X_2 + \ldots + X_k] = E[X_1] + E[X_2] + \ldots + E[X_k]$$

Let $X$ = sum of outcomes of two dice.
Let $X_i$ = the outcome of the $i^{th}$ dice.
What is the relationship of $X, X_1, and X_2$?
Can we compute $E[X]$?
Let $X = \text{sum of outcomes of two dice.}$
Let $X_i = \text{the outcome of the } i^{th} \text{ dice}$

$\Rightarrow X = X_1 + X_2$

$\Rightarrow E[X] = E[X_1+X_2] = E[X_1] + E[X_2]$

$= \frac{7}{2} + \frac{7}{2} = 7$

Can you compute the expectation of the sum of outcomes of three dice?