CS2351
Data Structures

Lecture 13:
Binary Search Tree

About this lecture

- A binary search tree (BST) is a binary tree that stores a set of items, and each item has a distinct key chosen from an ordered set
    - allows various queries and updates

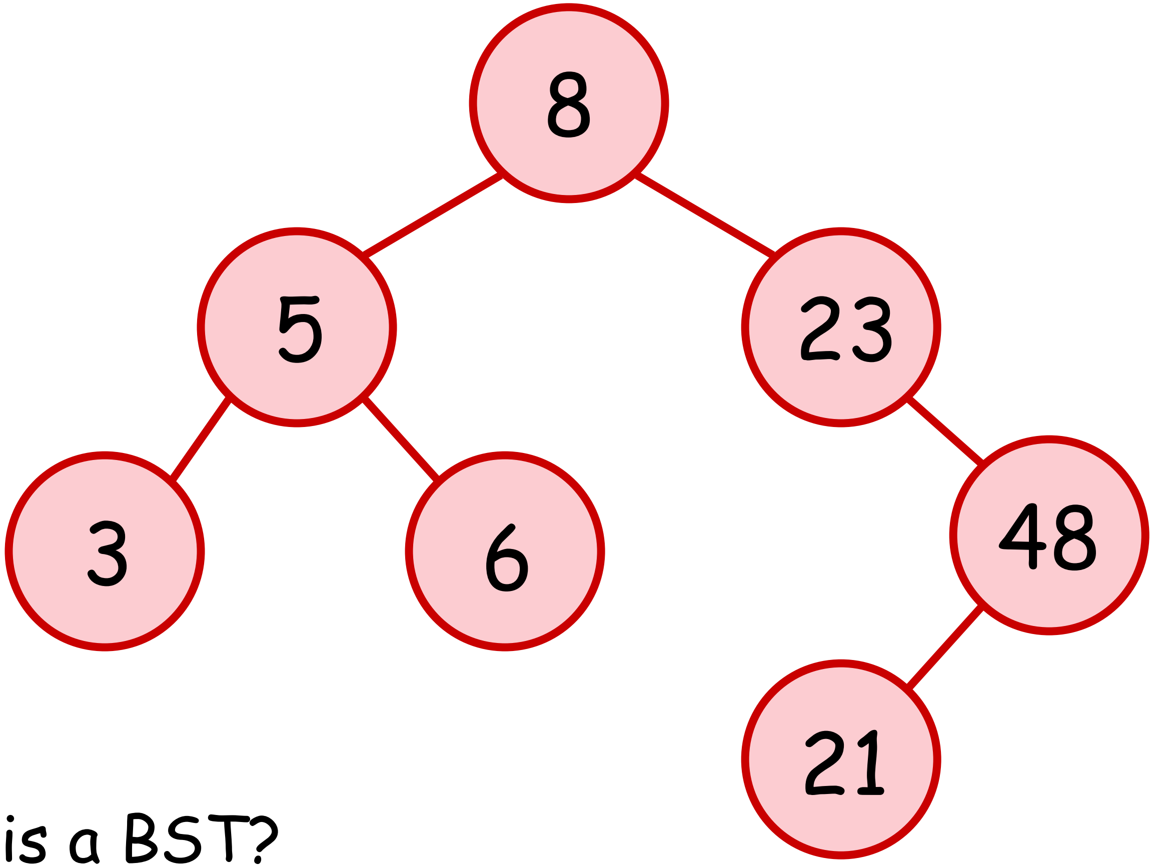
- In this lecture, we discuss how the BST supports the queries and the updates

Binary Search Tree (BST)

- Each node in a BST has a distinct key
- The keys in the nodes satisfies the following BST property :

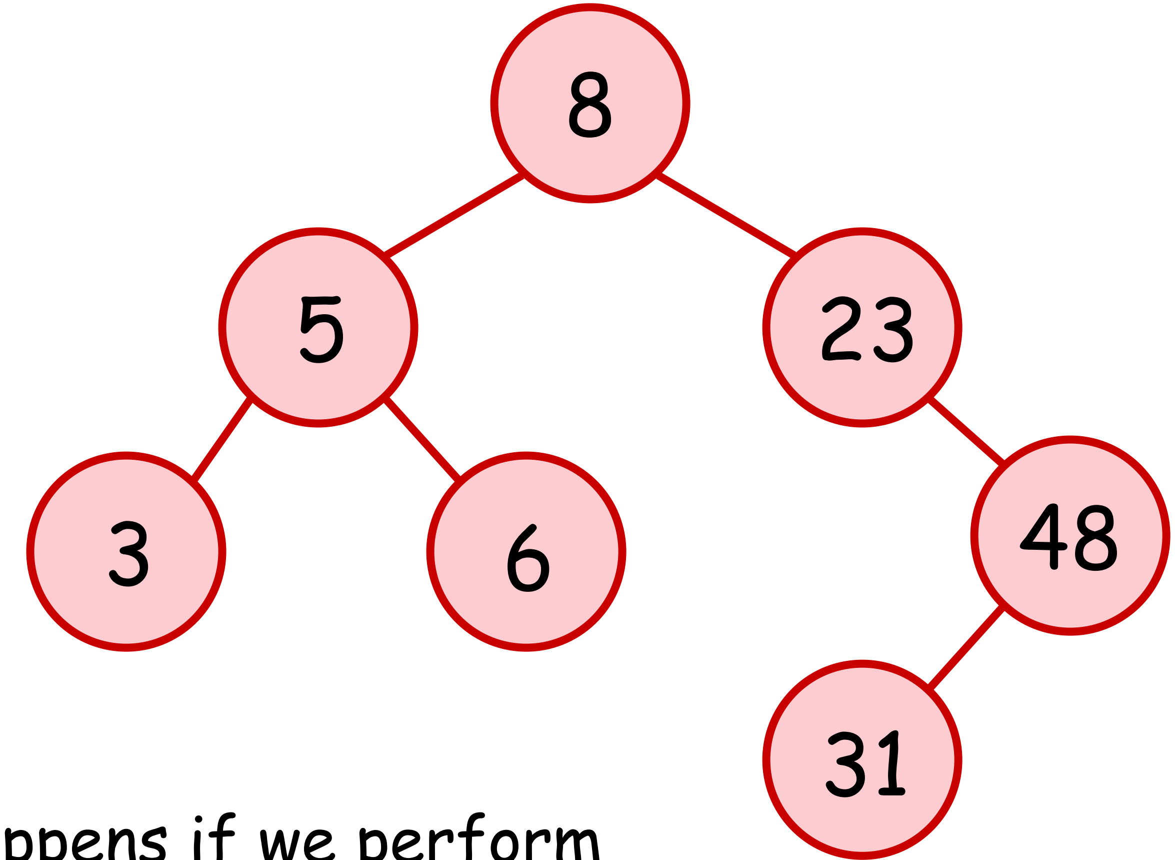
> Let x be a node in a BST.
>
> Let y and z be nodes in the left and right subtrees of x, respectively.
>
> Then we have y.key < x.key < z.key

Example of BST



Is this a BST?

Example of BST



What happens if we perform inorder traversal in a BST?

Queries in a BST

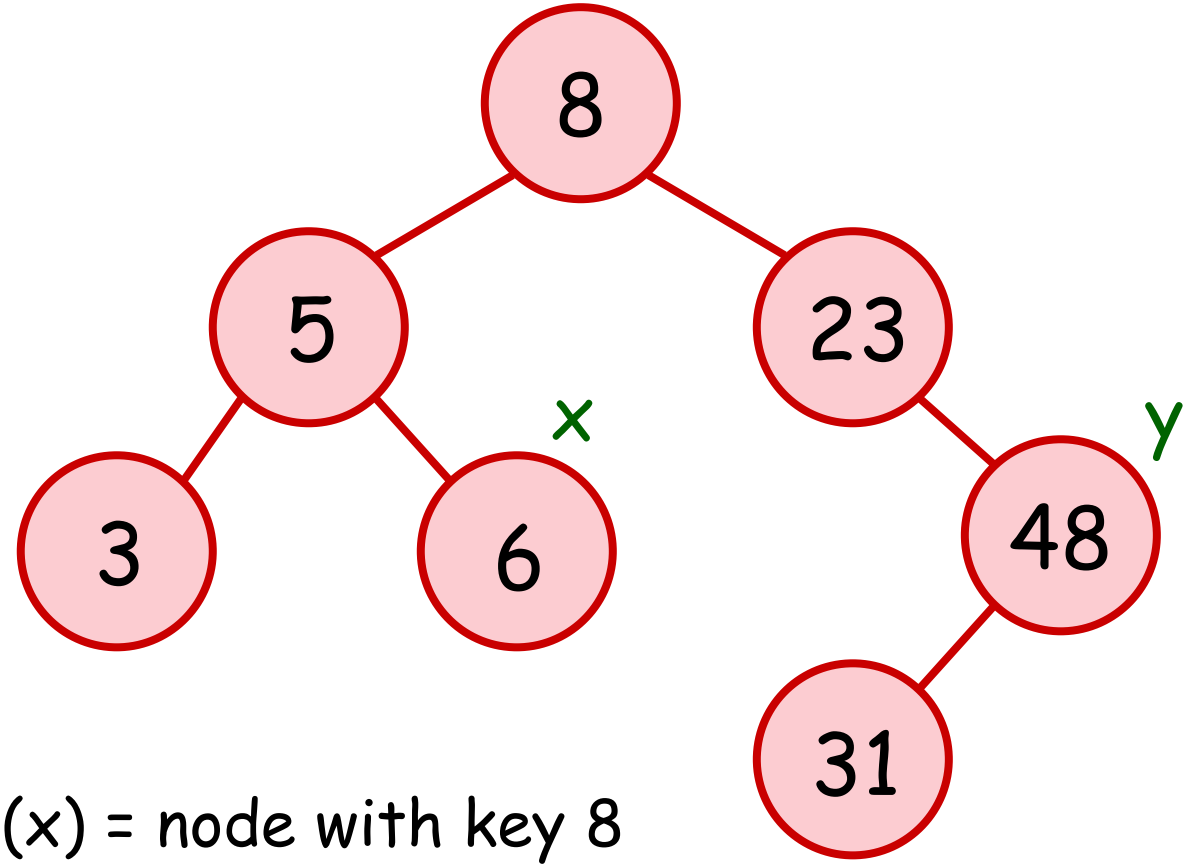
Queries in a BST

- A BST supports the following queries:
  1. Finding nodes with min or max keys
  2. Given a value k, search for a node that contains k as the key
  3. Given a node x, return the successor or the predecessor of x

successor: node with key just larger than x.key

predecessor: node with key just smaller than x.key

Successor and Predecessor



Successor(x) = node with key 8

Predecessor(y) = node with key 31

Finding Min or Max

- Where is the node with min key ?

  ➔ The leftmost node in BST

- Where is the node with max key ?

  ➔ The rightmost node in BST

- In general, let x be a node in the BST

  Q: Where is the node with min/max key in the subtree rooted at x ?

Implementation in C

- We define a function Min, which returns a pointer to the min key node in subtree of x

```
Node * Min( Node *x ) {
  while ( x->left != NULL )
     x = x->left ;
  return x ;
}
```

- Then desired min is equal to Min(r), where r = a pointer to the root of BST

Implementation in C

- We define a function Max, which returns a pointer to the max key node in subtree of x

```
Node * Max( Node *x ) {
  while ( x->right != NULL )
     x = x->right ;
  return x ;
}
```

- Then desired max is equal to Max(r)

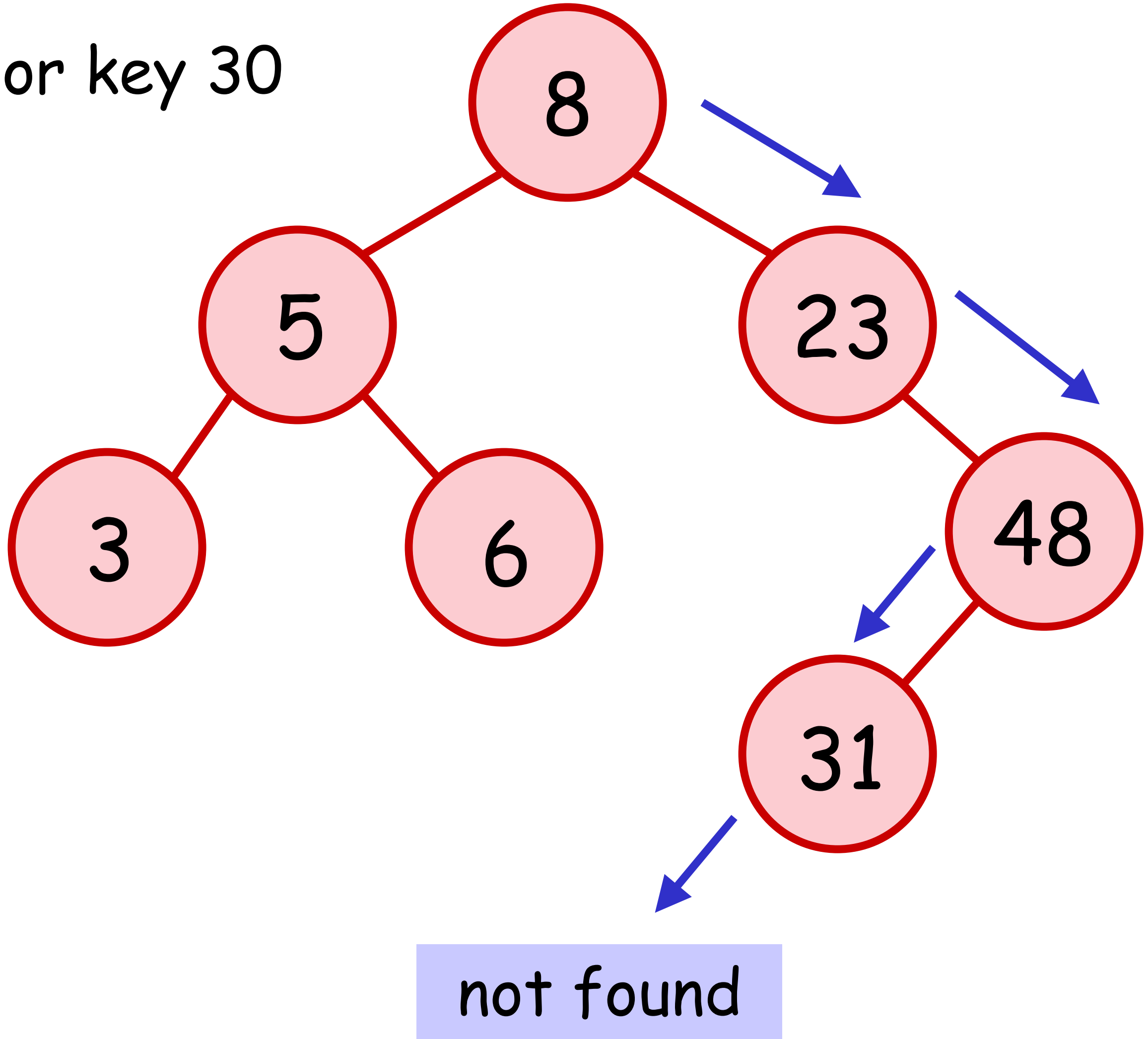
Searching a Key

- Let k be the key to be searched. Suppose k < root.key. What can we conclude ?
- In fact, searching a BST is very similar to doing binary search in a sorted array :

1. If k is equal to root.key, done !
2. Else if k < root.key, recursively search left subtree of root
3. Else, recursively search right subtree

Example of Searching a BST

Search for key 30



Implementation in C

- We define a function Search :

```
Node * Search( Node *x, int k ) {
  if ( x == NULL )   return NULL ;
  if ( x->key == k ) return x ;
  if ( x->key > k )
    return Search( x->left, k );
  return Search( x->right, k ) ;
}
```

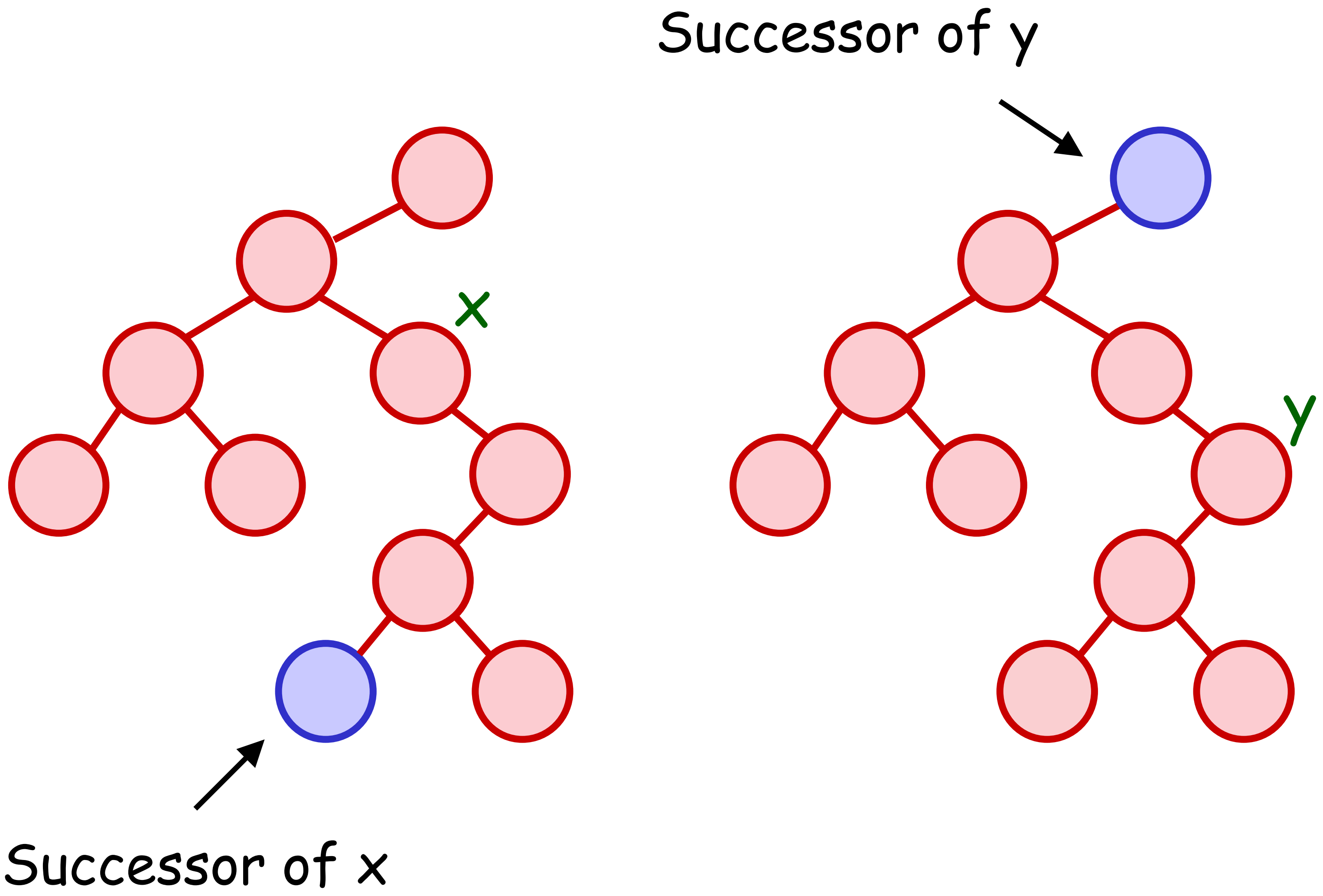
- Then, desired node = Search(r, k) , where r = pointer to root of BST

Finding Successor

- Let x be a node in the BST
- The successor of x is the next node in the inorder traversal

1. What if x has a right child ?
   ➔ min in the subtree of right child
2. What if not ?
   ➔ first ancestor "on the right" of x

Finding Successor



Implementation in C

- To help finding successor, we assume that each node has a parent pointer
- Then we can define Successor as follows :

```
Node * Successor( Node *x ) {
  if ( x->right != NULL )
      return Min( x->right ) ;
  y = x->parent ;
  while ( y != NULL && x == y->right )
  {    x = y ;  y = y->parent ;  }
  return y ;
}
```

Implementation in C

- Similarly, we can define Predecessor :

```
Node * Predecessor( Node *x ) {
  if ( x->left != NULL )
      return Max( x->left ) ;
  y = x->parent ;
  while ( y != NULL && x == y->left )
  {    x = y ;  y = y->parent ;  }
  return y ;
}
```

Query Performance

- Let h denote the node-height of the BST

> Theorem:
>
> The queries minimum, maximum, search, predecessor, and successor can each be performed in O(h) time

- What is the value of h in the best case ? How about the worst case ?

Updates in a BST

Updates in a BST

- A BST supports the following updates:
  1. Inserting a node z with key k
  2. Deleting a node x

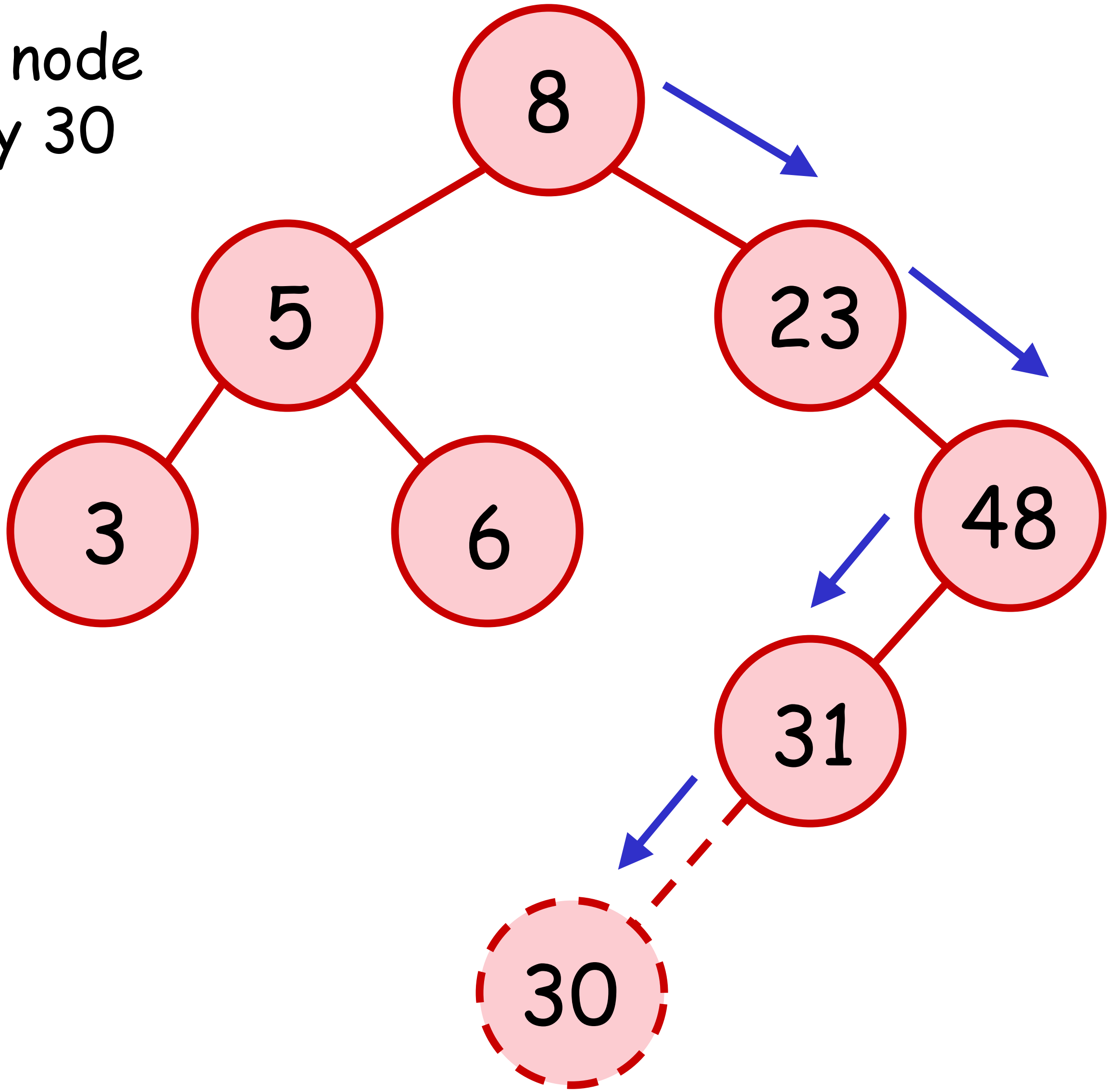
- Note: When we perform updates, we have to maintain the BST property

Inserting a Node

- Let z be a new node to be inserted, and k be its key
- Observation : After insertion, k becomes searchable in BST
  - ➔ the insertion position is the same as the position we expect to find k
- Insertion is done by slightly modifying the searching algorithm

Example of Insertion in BST

Insert a node
with key 30



Implementation in C

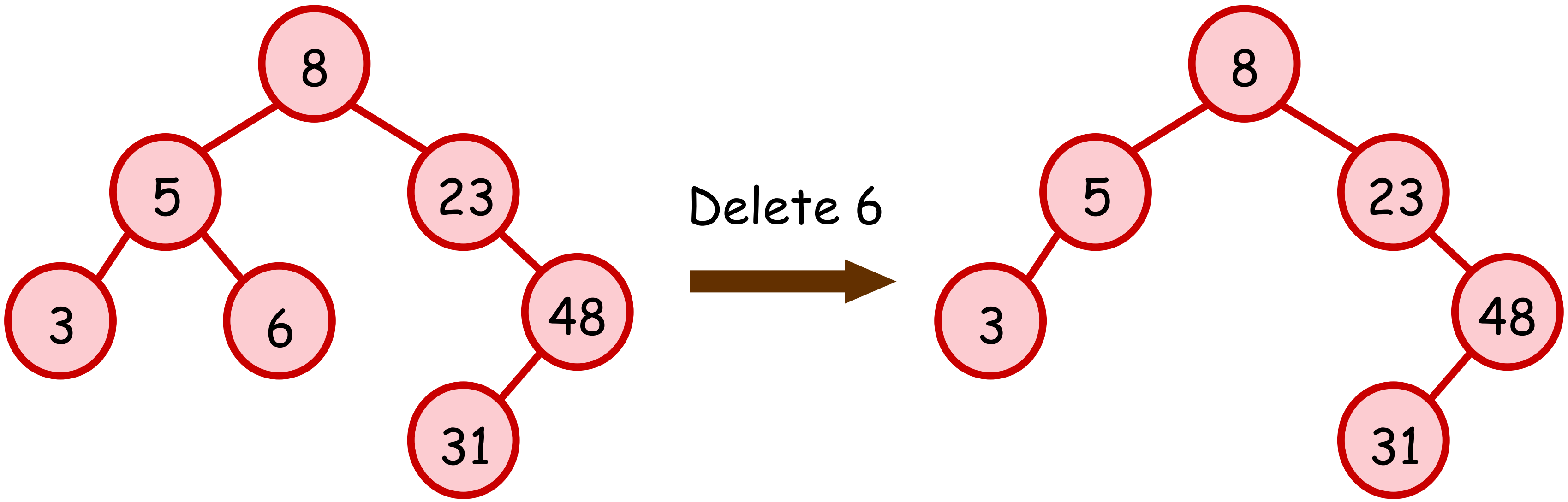
```
void Insert( Node *x, Node *z ) {
  if ( x->key > z->key ) {
    if ( x->left )  Insert( x->left, z );
    else x->left = z ;
  }
  else if ( x->key < z->key ) {
    if ( x->right ) Insert( x->right, z );
    else x->right = z ;
  }
}
```

- Then, insertion is done by Insert(r, z) , where r = pointer to root of BST

Deleting a Node

- Let x be a node to be deleted
- Case 1 :

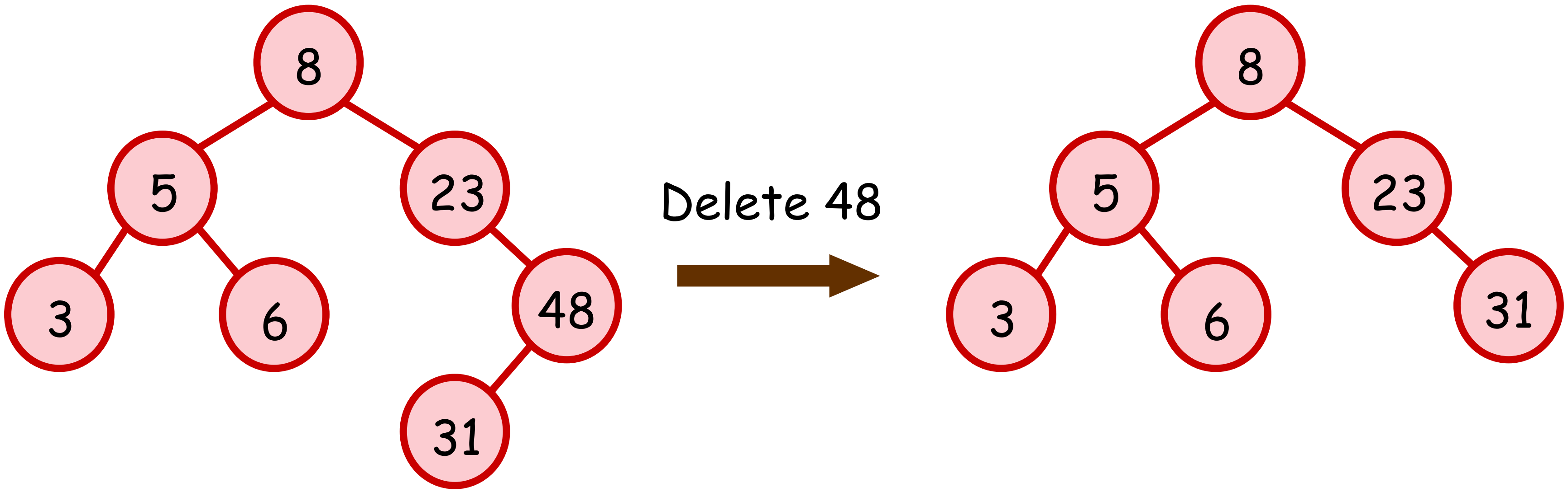
  If x is a leaf, we just remove x



Deleting a Node

- Case 2 :

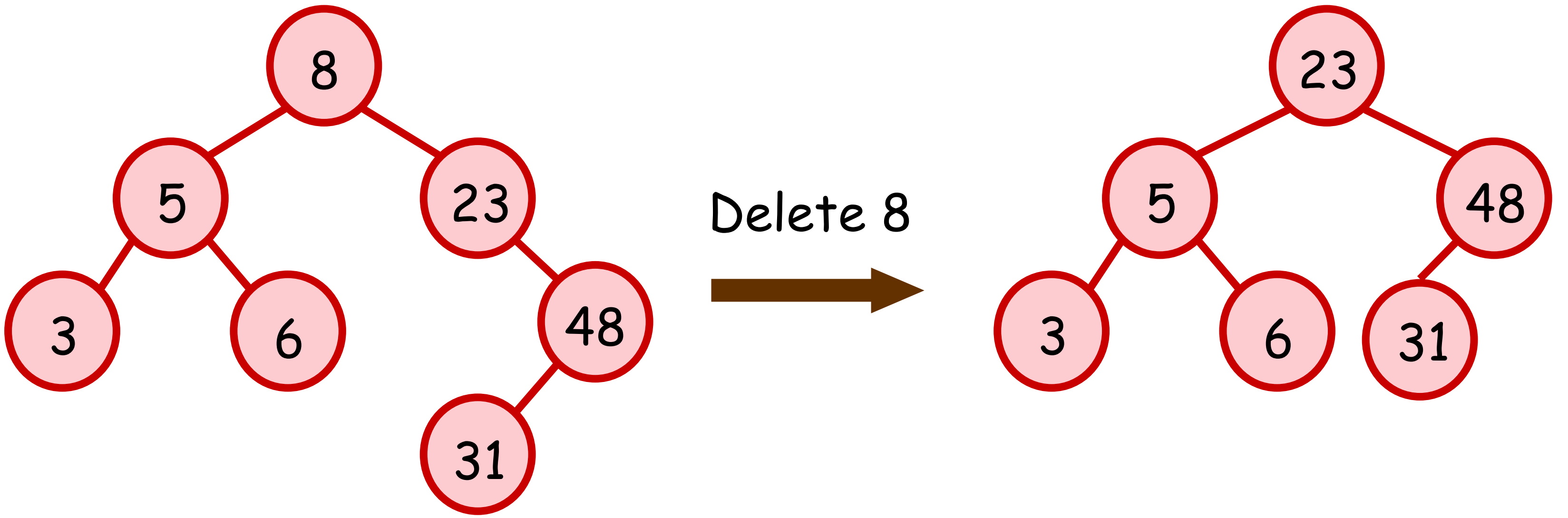
  If x has one child, we connect x's parent to its child



Deleting a Node

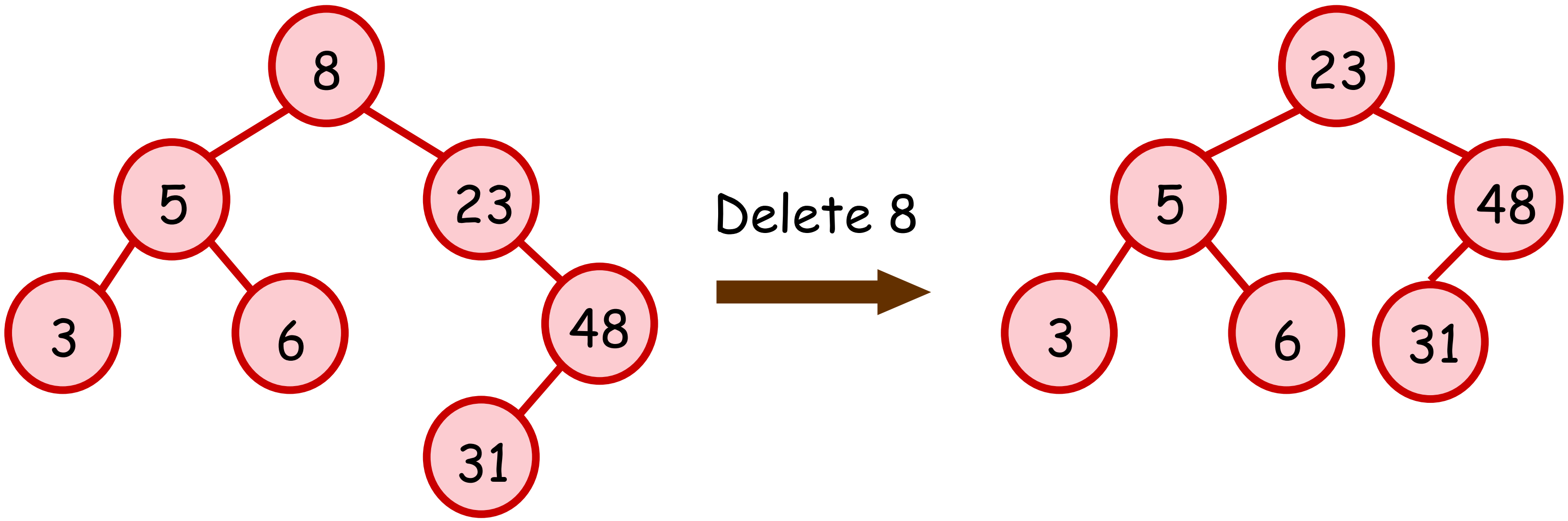
- Case 3 :

  If x has two children, we swap x with its successor, and then delete x



Deleting a Node

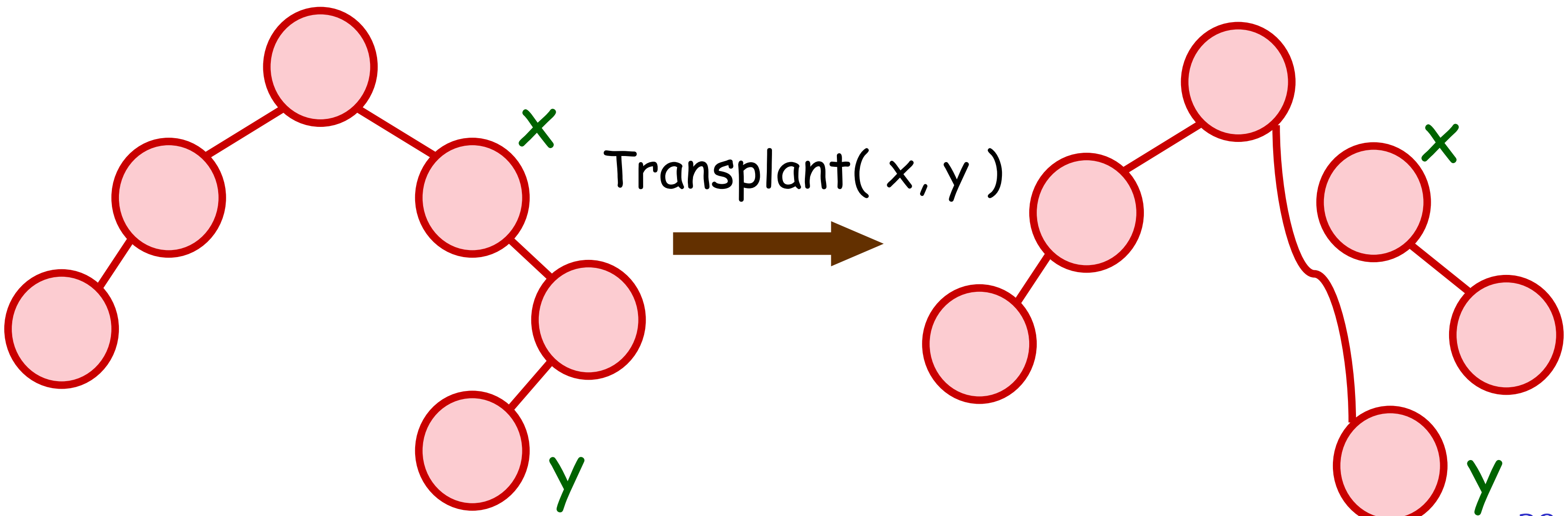
- In Case 3, the successor of x does not have a left child. Why?



Implementation in C

- To ease our discussion, we now define a function Transplant, such that :

  Transplant(x, y) links x's parent to y and y's parent is changed accordingly



Implementation in C

- The function Transplant(x, y) can be easily implemented as follows :

```
void Transplant( Node *x, Node *y ) {
  if ( x->parent == NULL )   // x is root
  {   r = y ; }              // set y as root
  else if ( x->parent->left == x )
  {   x->parent->left = y ;  }
  else {   x->parent->right = y ;  }
  if ( y != NULL )
     y->parent = x->parent ;
}
```

Implementation in C

- Now Case 1 can be implemented as follows :

```
void Delete( Node *x ) {

  /* Case 1:  x is a leaf */
  if ( !x->left && !x->right )
     Transplant( x, NULL );

  /* Case 2 and Case 3 */
  ...
}
```

Implementation in C

… and Case 2 can be implemented as follows :

```
void Delete( Node *x ) {
  /* Case 1 */ ...
  /* Case 2:  x has one child */
  else if ( x->left == NULL )
     Transplant( x, x->right ) ;
  else if ( x->right == NULL )
     Transplant( x, x->left ) ;
  /* Case 3 */ ...
}
```

Implementation in C

- For Case 3, we have two subcases :

```
void Delete( Node *x ) {
  /* Case 1 and Case 2 */ ...
  else {  /* Case 3 : x has two children */
    y = Min( x->right );  // get successor
    if ( y->parent == x ) { // Subcase 3.1
      Transplant( x, y ) ; y->left = x->left ;
      x->left->parent = y ;
    }
    else { /* Subcase 3.2 */ ... }
  }
}
```

Implementation in C

```
void Delete( Node *x ) {
  else { /* Case 3: x has two children */
    ...
    else { // Subcase 3.2
      Transplant( y, y->right ) ;
      Transplant( x, y ) ;
      y->right = x->right; y->left = x->left;
      x->right->parent = x->left->parent = y;
    }
  }
}
```

Update Performance

- Let h denote the node-height of the BST

Theorem:

Inserting or deleting a node in a BST can each be performed in $O(h)$ time

Remarks

- The implementation here discusses the core idea, and does not handle the boundary cases well

  Ex: insertion in an empty BST, or
  deletion resulting an empty BST

- Also, more than one way to implement

  Ex: deletion can be done by swapping with the predecessor, search can be done with while-loop instead of recursion