CS2351
Data Structures

Lecture 10:
Graph and Tree Traversals I

About this lecture

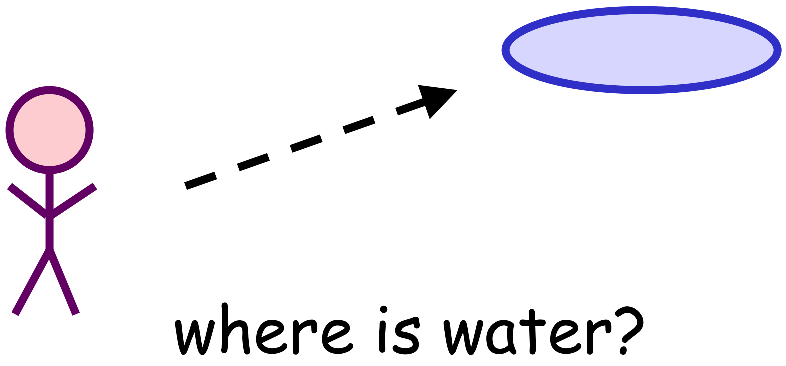
- We introduce two popular algorithms to traverse a graph

  1. Breadth First Search (BFS)
  2. Depth First Search (DFS)
     - DFS Tree and DFS Forest
     - Parenthesis theorem

Breadth First Search

Lost in a Desert

- After an unfortunate accident, we survived, but are lost in a desert
- To keep surviving, we need to find water
- How to find the closest water source?



Breadth First Search (BFS)

- A simple algorithm to find all vertices reachable from a particular vertex s
  - s is called source vertex
- Idea: Explore vertices in rounds
  - At Round k, visit all vertices whose shortest distance (#edges) from s is k-1
  - Also, discover all vertices whose shortest distance from s is k

The BFS Algorithm

1. Mark s as discovered in Round 0
2. For Round k = 1, 2, 3, …,

   For (each u discovered in Round k-1)

   { Mark u as visited ;

   Visit each neighbor v of u ;

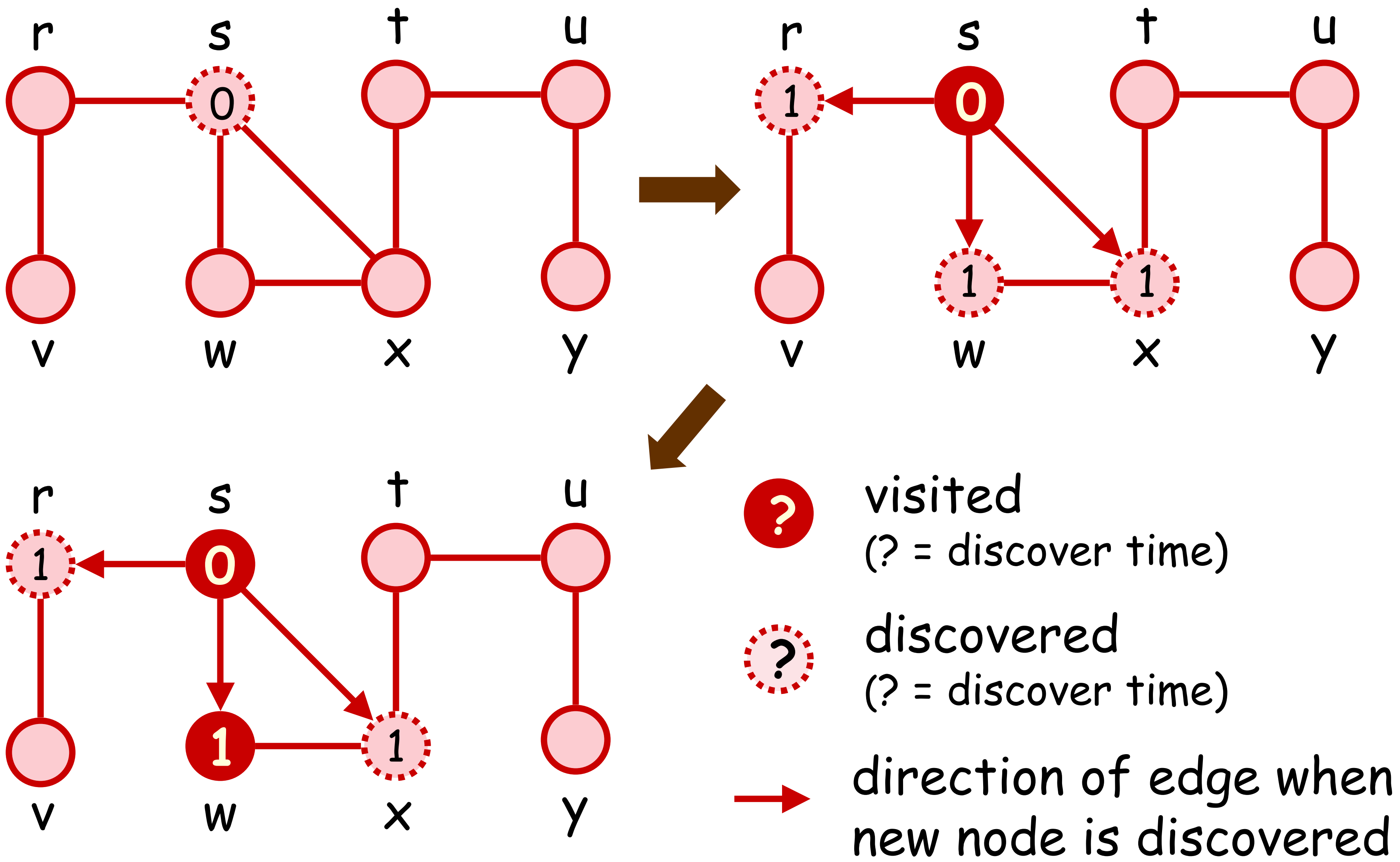
   If (v not visited and not discovered)

   Mark v as discovered in Round k ;

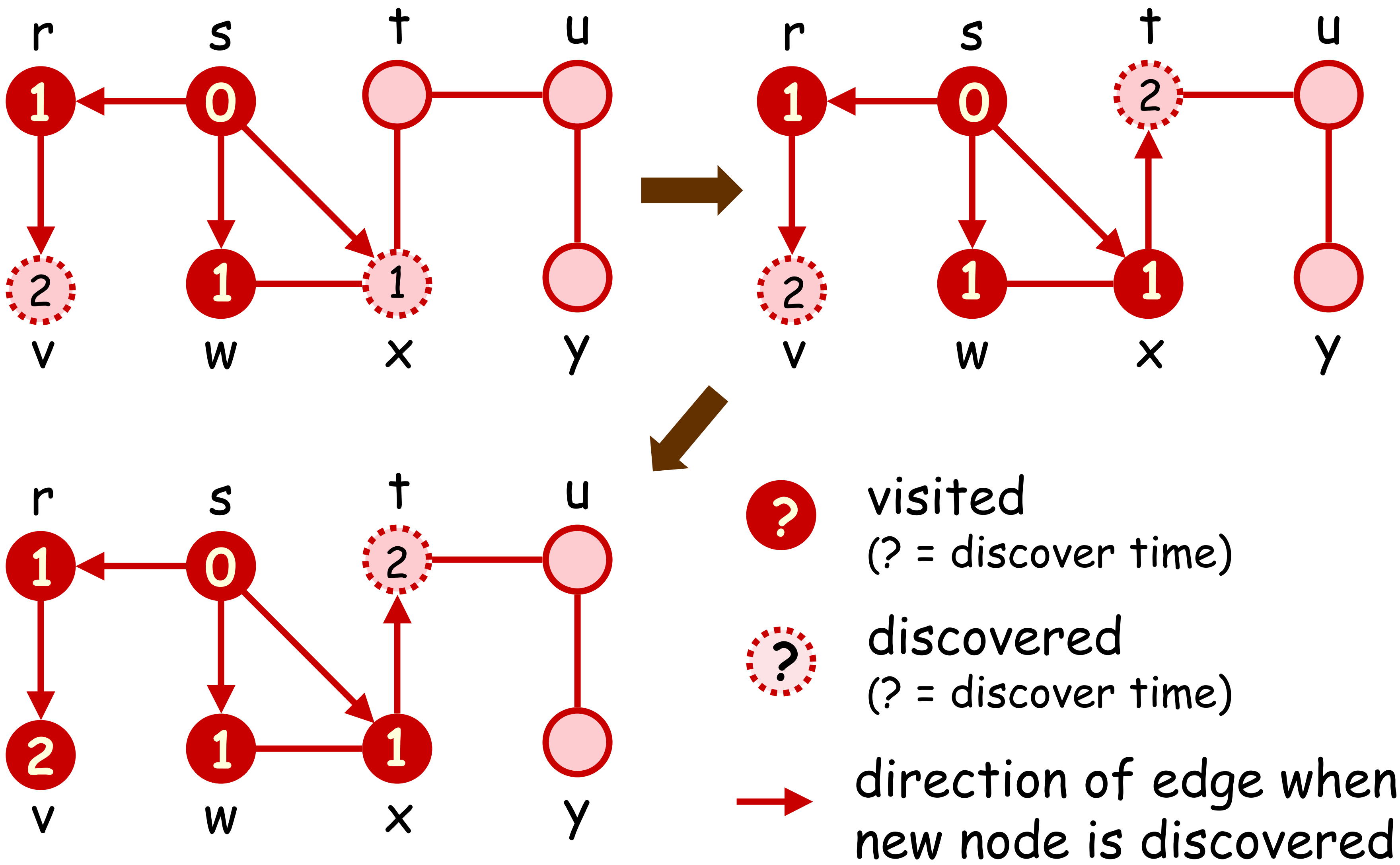
   }

Stop if no vertices were discovered in Round k-1

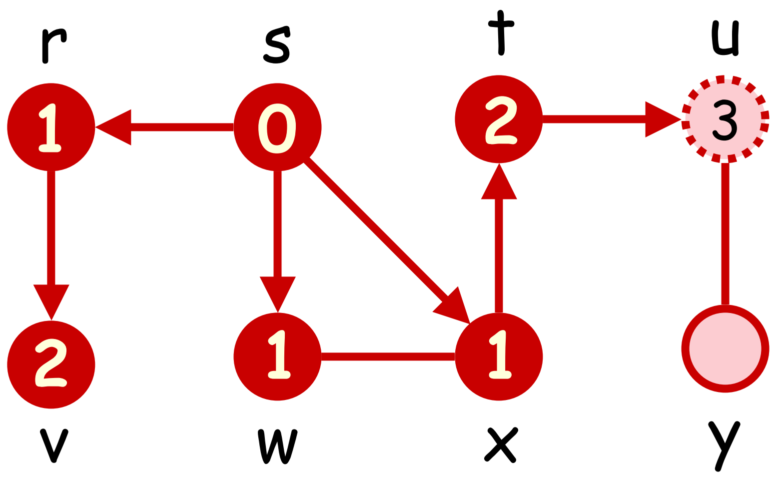
Example (s = source)

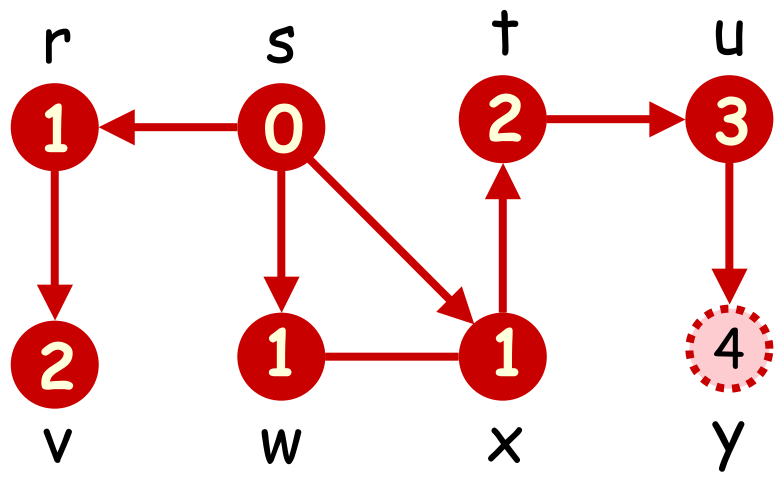


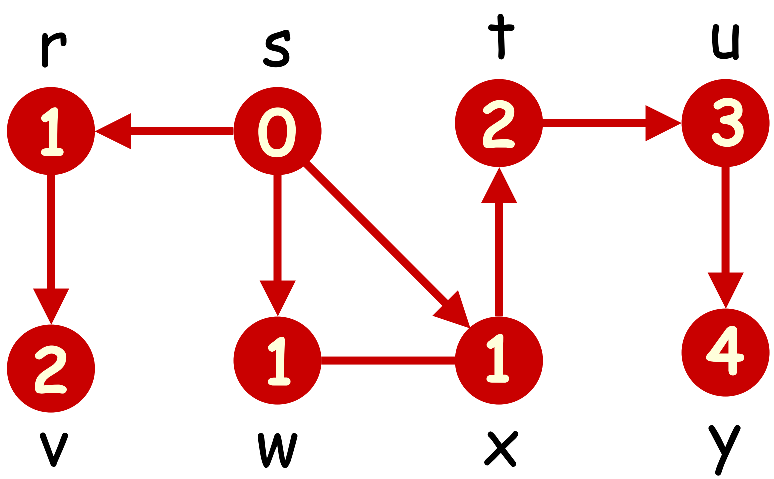
Example (s = source)



Example (s = source)





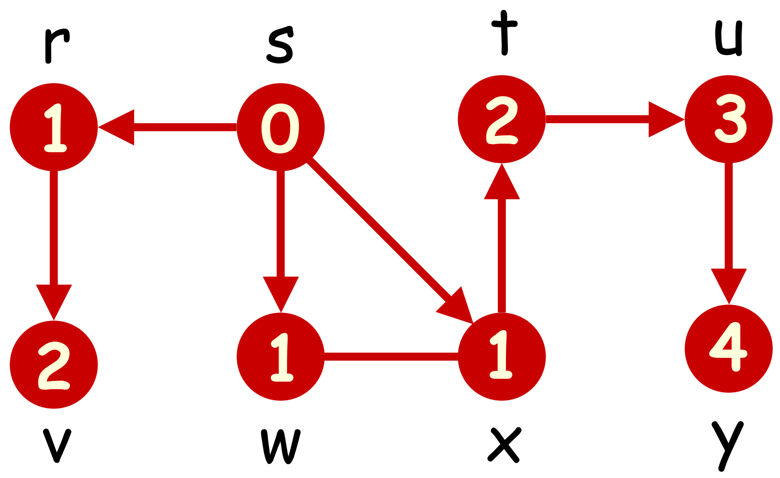


? visited
(? = discover time)

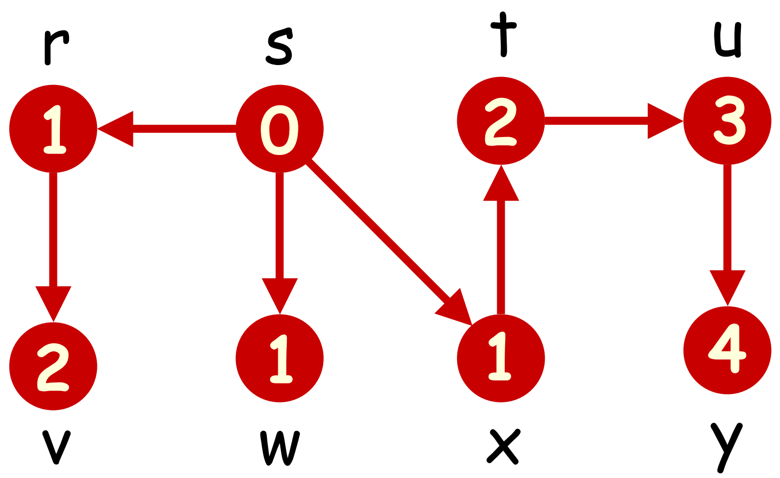
? discovered
(? = discover time)

→ direction of edge when new node is discovered

Example (s = source)



Done when no new node is discovered



The directed edges form a tree that contains all nodes reachable from s

Called BFS tree of s

Correctness

- The correctness of BFS follows from the following theorem :

Theorem: A vertex v is discovered in Round k if and only if shortest distance of v from source s is k

Proof: By induction

Performance

- BFS algorithm is easily done if we use
  - an O(|V|)-size array to store discovered/visited information
  - a separate list for each round to store the vertices discovered in that round
- Since no vertex is discovered twice, and each edge is visited at most twice (why?)

➔ Total time: O(|V|+|E|)

➔ Total space: O(|V|+|E|)

Performance (2)

- Instead of using a separate list for each round, we can use a common queue
  - When a vertex is discovered, we put it at the end of the queue
  - To pick a vertex to visit in Step 2, we pick the one at the front of the queue
  - Done when no vertex is in the queue

➔ No improvement in time/space …

➔ But algorithm is simplified

Question: Can you prove the correctness of using queue?

Depth First Search

Depth First Search (DFS)

- An alternative algorithm to find all vertices reachable from a particular source vertex s
- Idea:

  Explore a branch as far as possible before exploring another branch

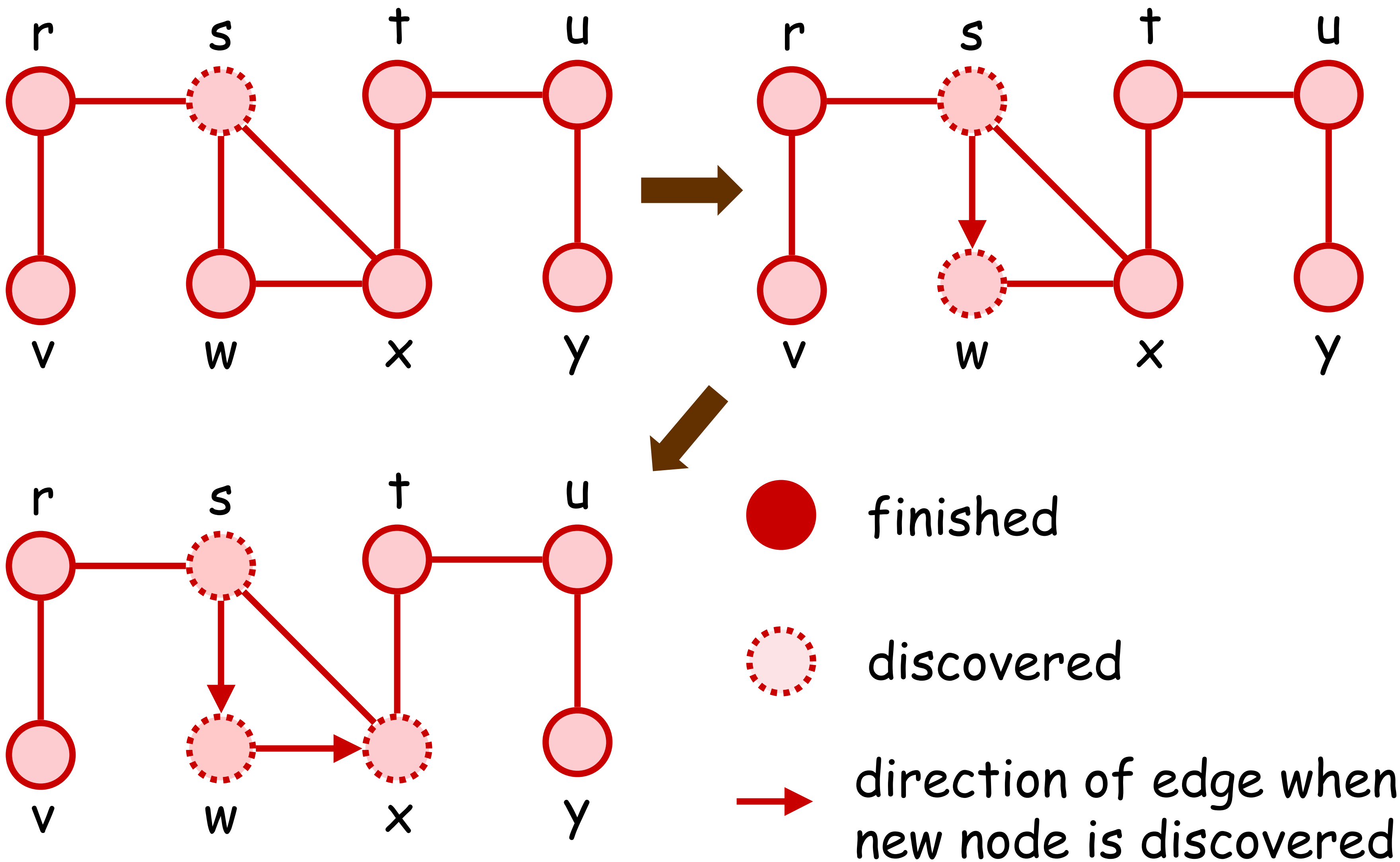
- Easily done by recursion or stack

The DFS Algorithm

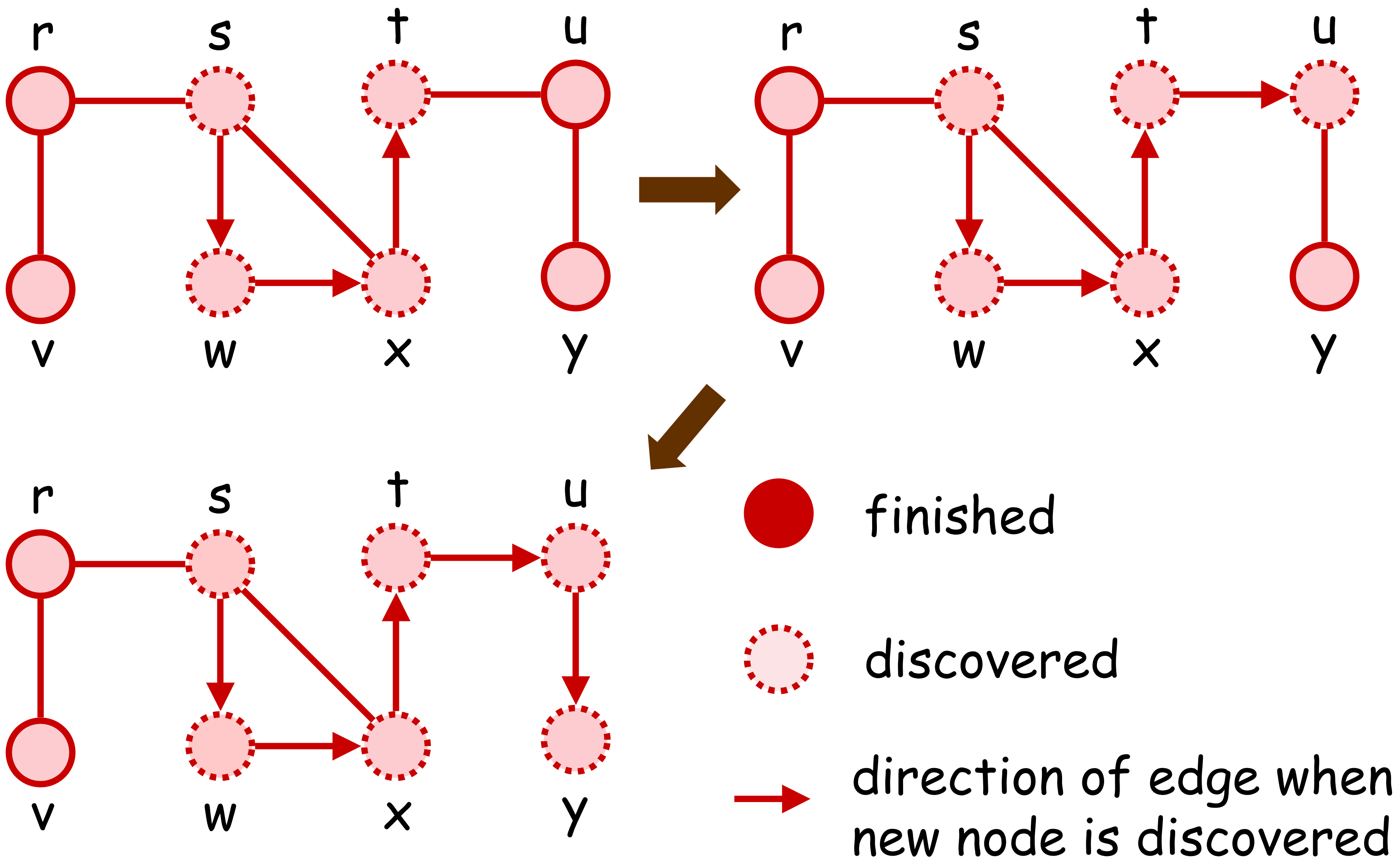
```
DFS(u)
{    Mark u as discovered ;
     while (u has unvisited neighbor v)
            DFS(v);
     Mark u as finished ;
}
```

The while-loop explores a branch as far as possible before the next branch

Example (s = source)



Example (s = source)



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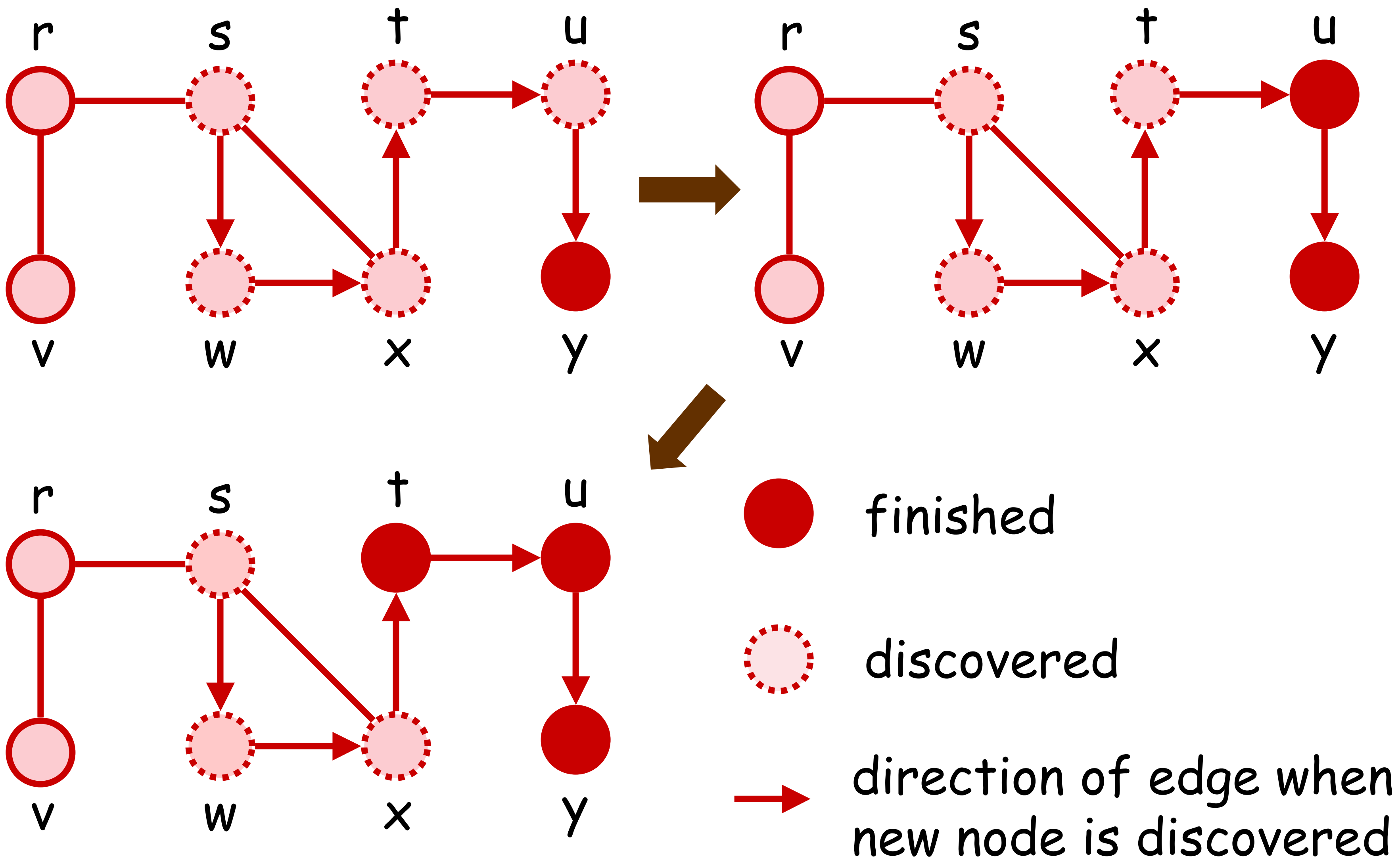

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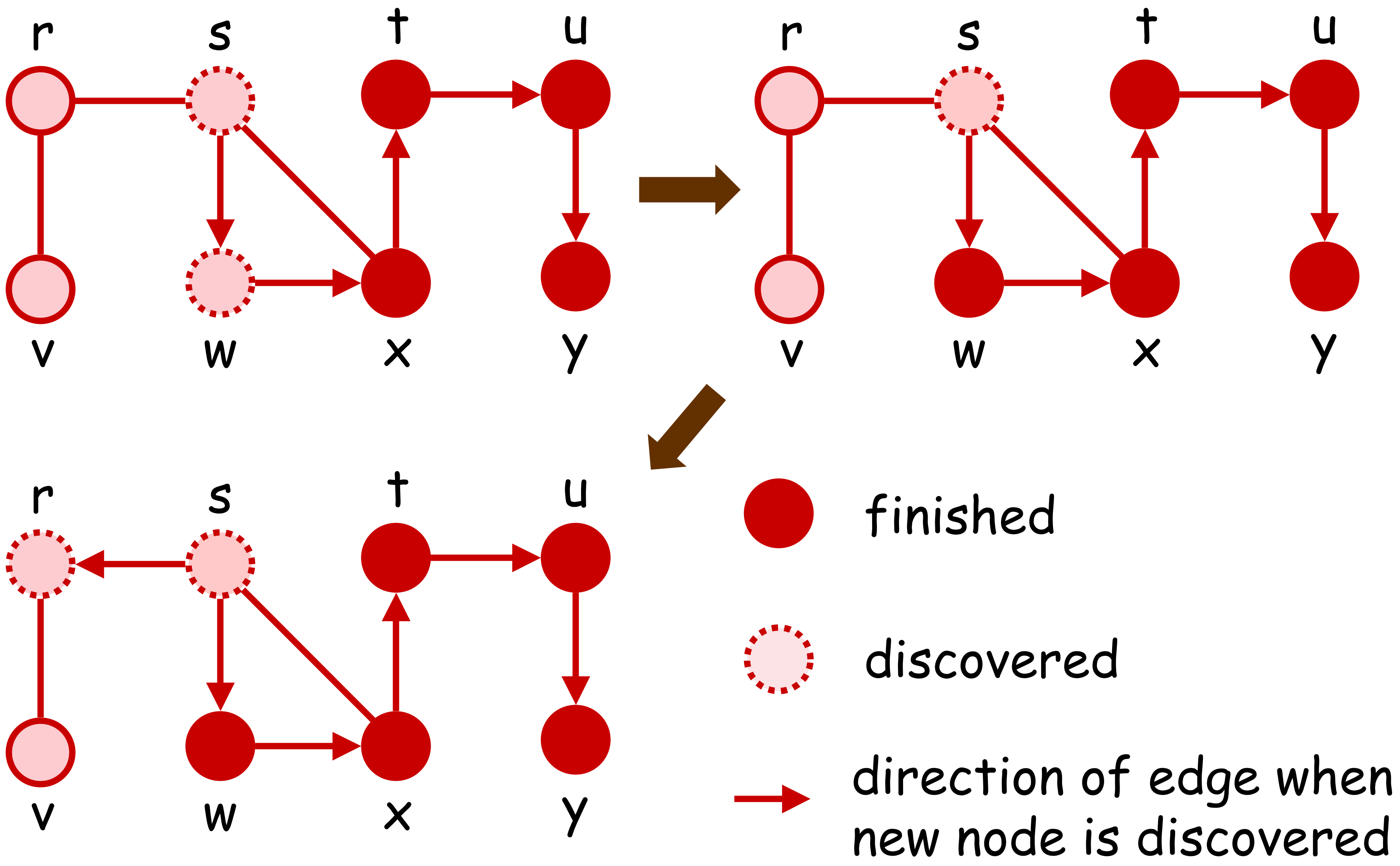
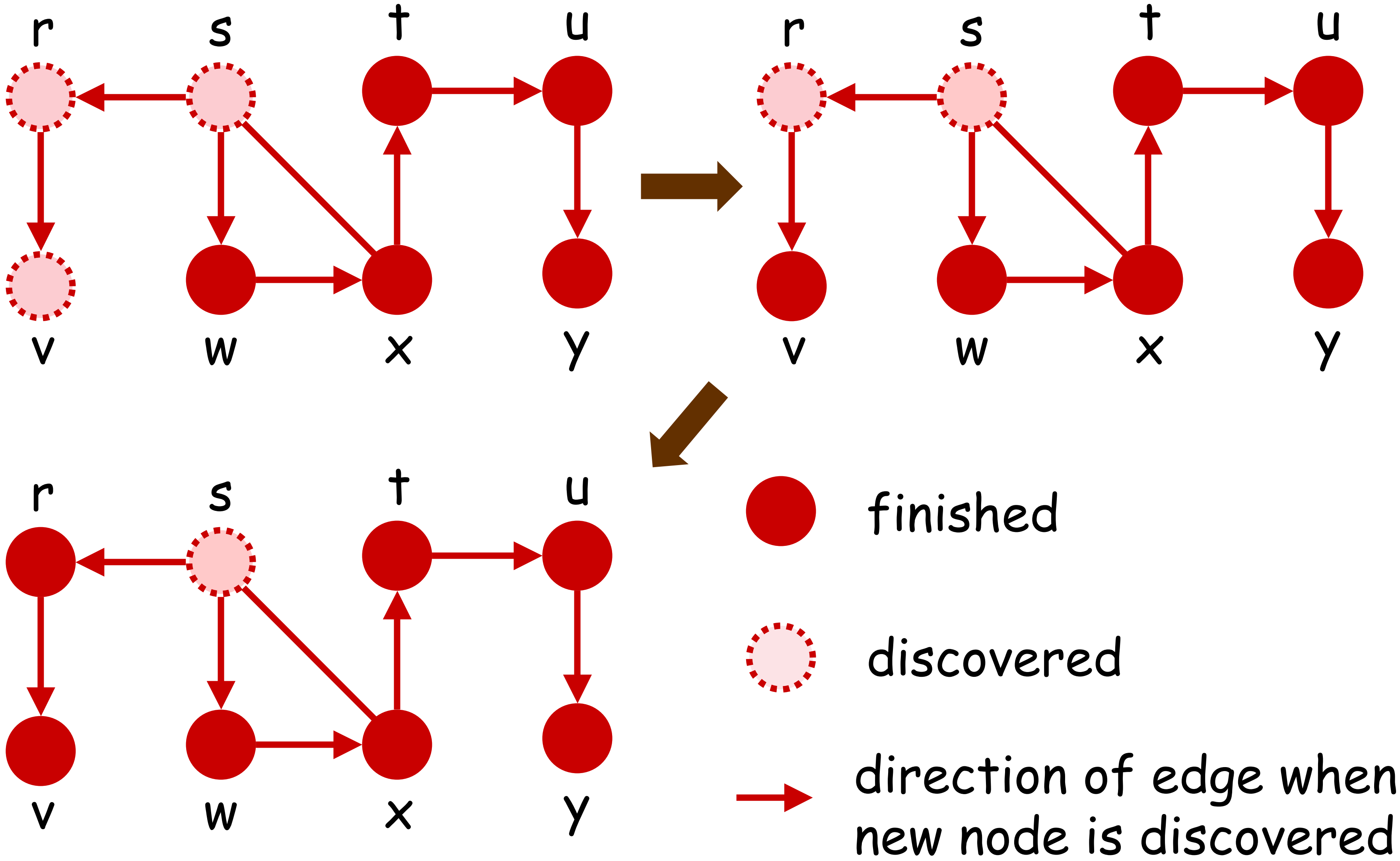
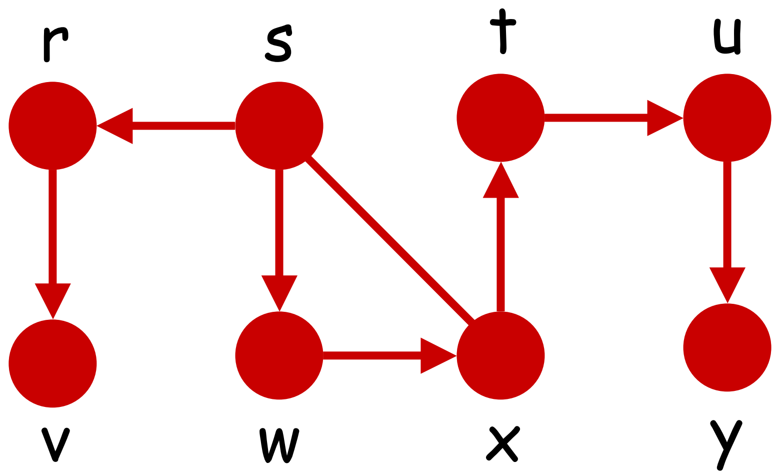


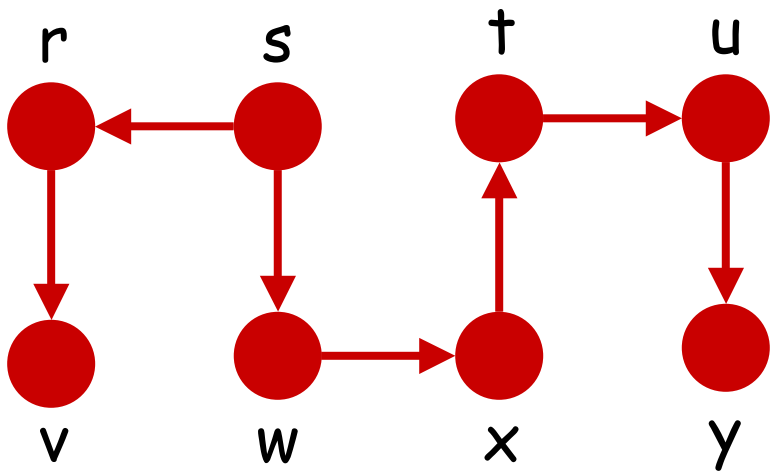
Example (s = source)



Example (s = source)



Done when s is discovered



The directed edges form a tree that contains all nodes reachable from s

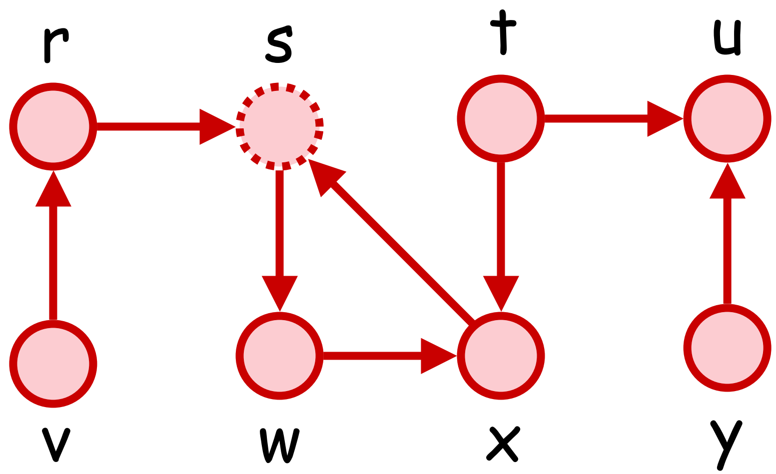
Called DFS tree of s

Generalization

- Just like BFS, DFS may not visit all the vertices of the input graph G, because :
  - G may be disconnected
  - G may be directed, and there is no directed path from s to some vertex
- In most application of DFS (as a subroutine) , once DFS tree of s is obtained, we will continue to apply DFS algorithm on any unvisited vertices …

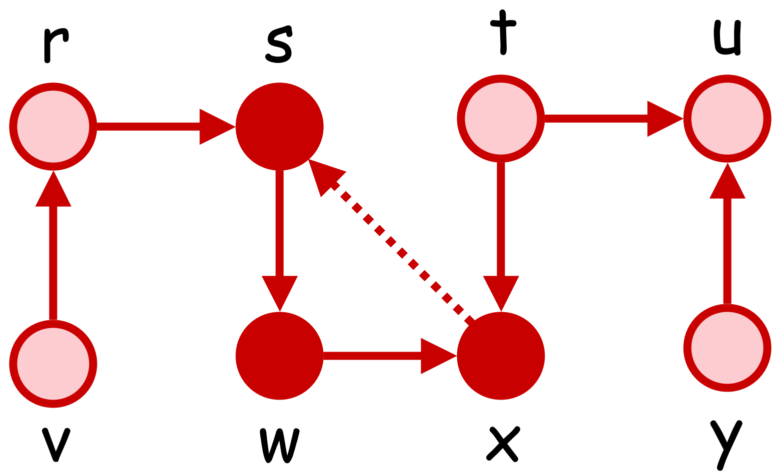
Generalization (Example)

Suppose the input graph is directed



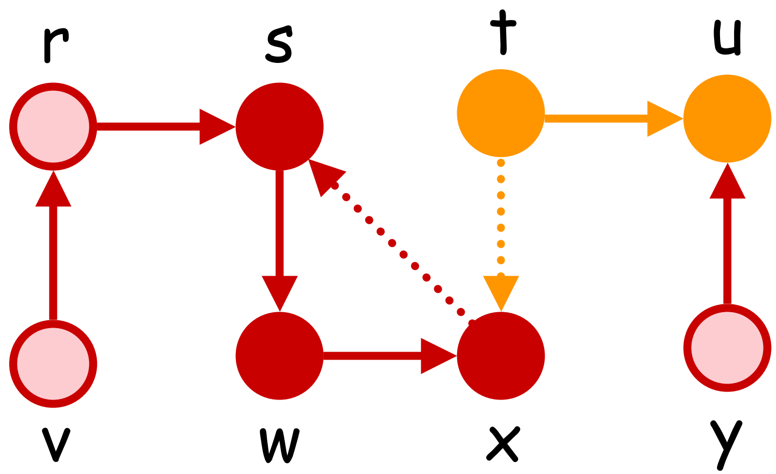
Generalization (Example)

1. After applying DFS on s



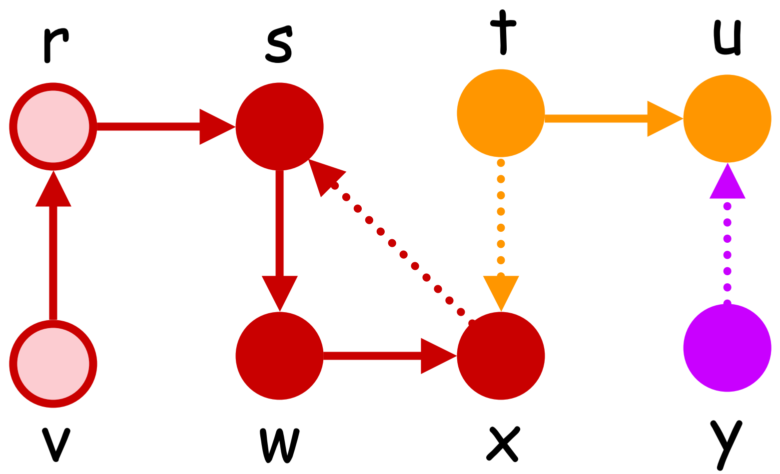
Generalization (Example)

2. Then, after applying DFS on t



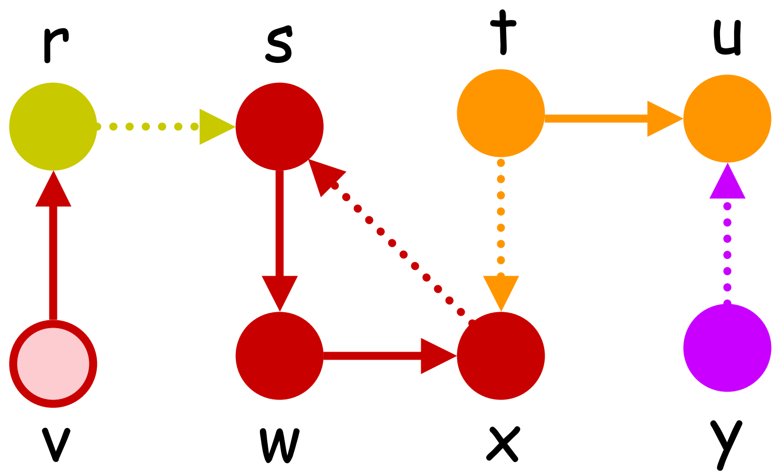
Generalization (Example)

3. Then, after applying DFS on y



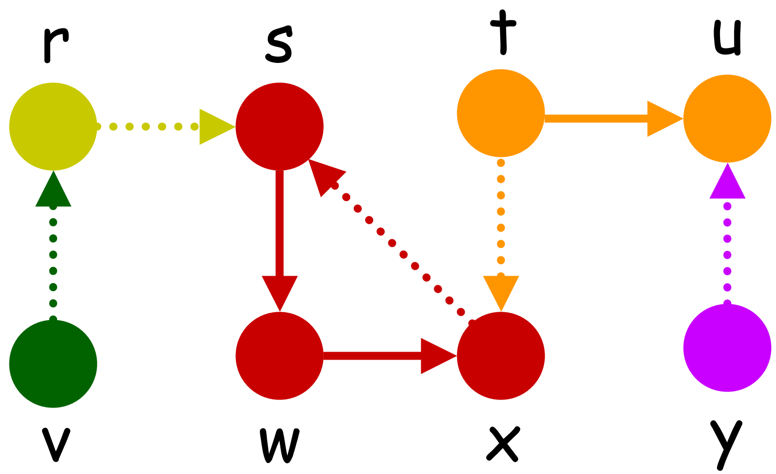
Generalization (Example)

4. Then, after applying DFS on r



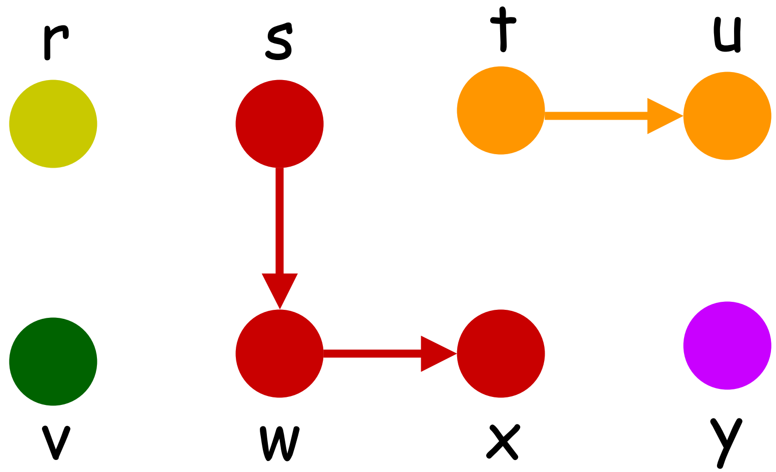
Generalization (Example)

5. Then, after applying DFS on v



Generalization (Example)

Result : a collection of rooted trees called DFS forest



Performance

- Since no vertex is discovered twice, and each edge is visited at most twice (why?)
  ➔ Total time: $O(|V|+|E|)$

- As mentioned, apart from recursion, we can also perform DFS using a LIFO stack (Do you know how?)

Discovery and Finishing Times

- When the DFS algorithm is run, let us consider a global time such that the time increases one unit :
  - when a node is discovered, or
  - when a node is finished

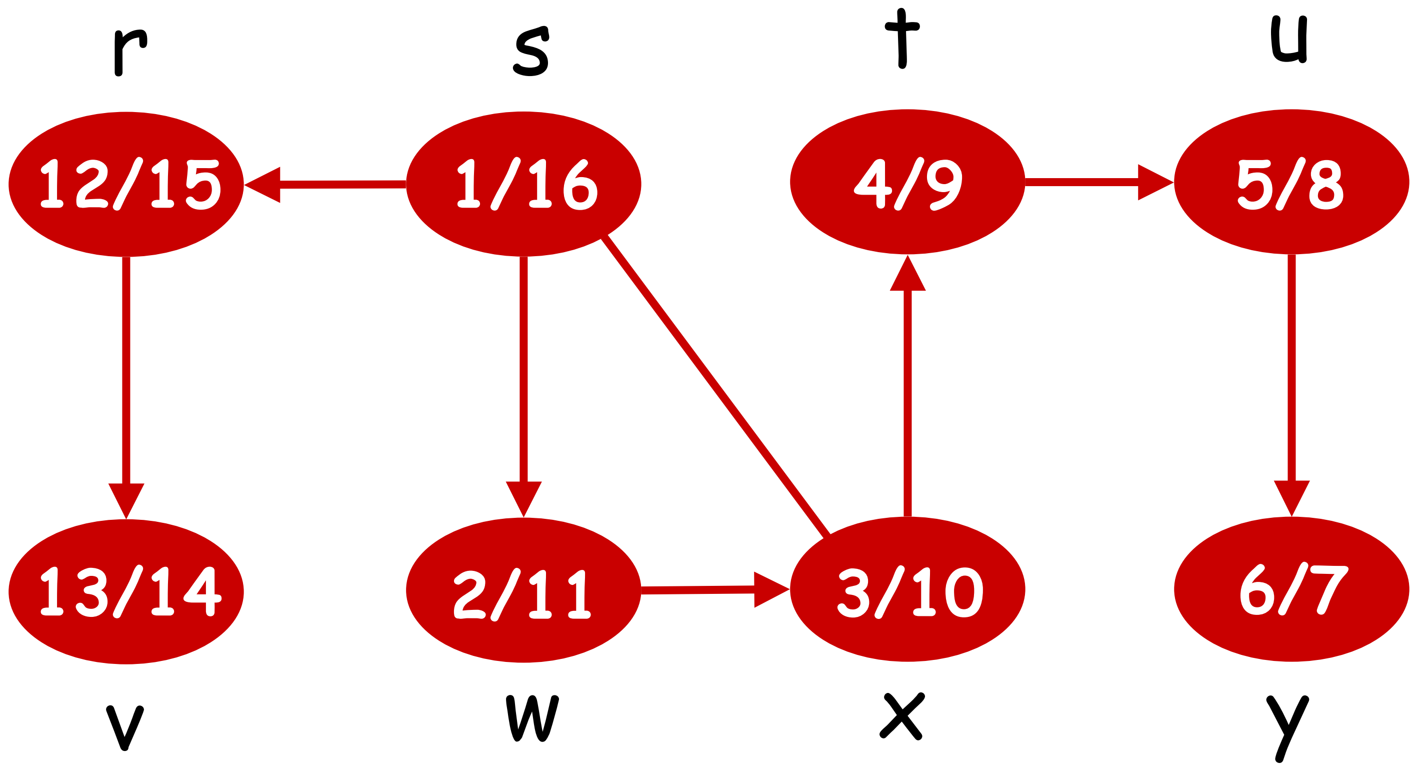
  (i.e., finished exploring all unvisited neighbors)

- Each node u records :

  d(u) = the time when u is discovered, and

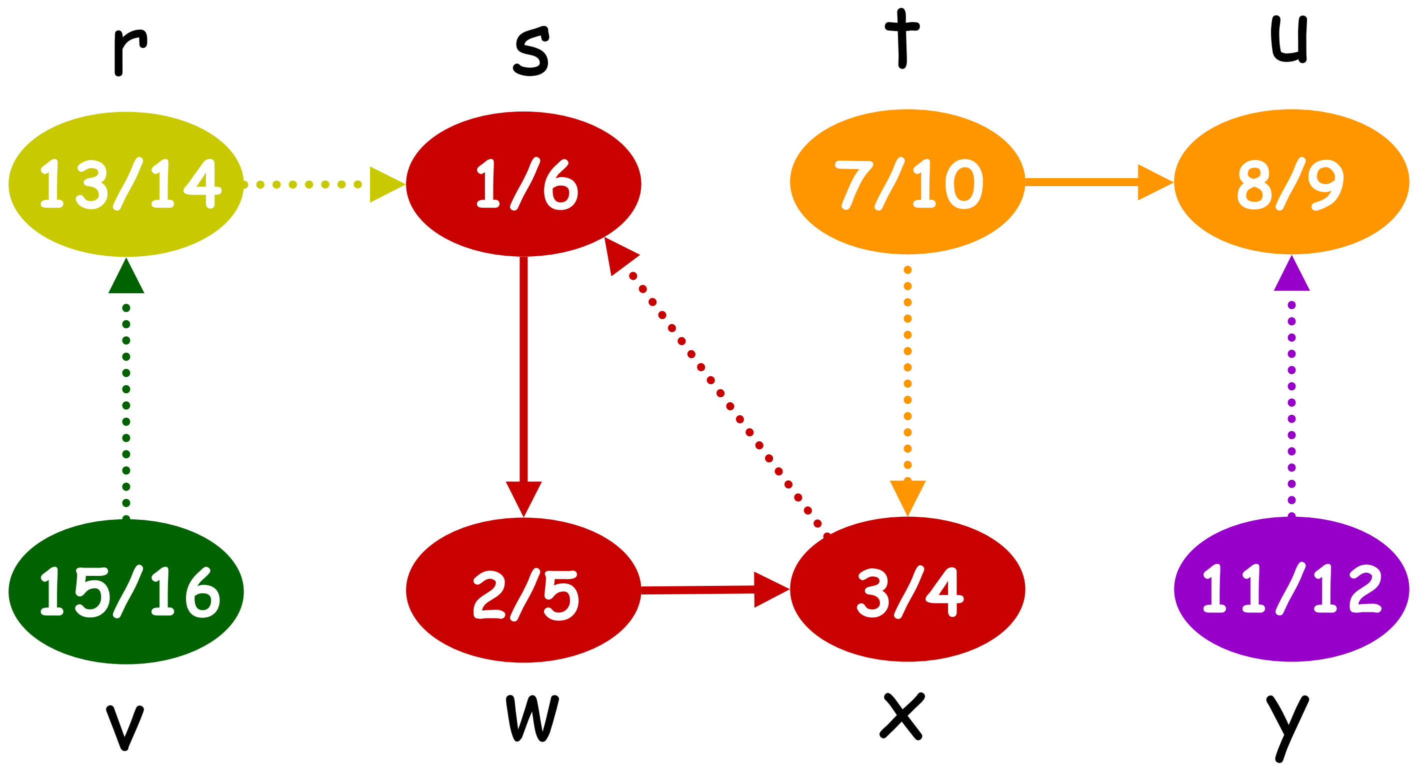
  f(u) = the time when u is finished

Discovery and Finishing Times



In our first example
(undirected graph)

Discovery and Finishing Times



In our second example
(directed graph)

Nice Properties

Lemma: For any node u, $d(u) < f(u)$

Lemma: For nodes u and v,
$d(u)$, $d(v)$, $f(u)$, $f(v)$ are all distinct

Theorem (Parenthesis Theorem):
Let u and v be two nodes with $d(u) < d(v)$.
Then, either

1. $d(u) < d(v) < f(v) < f(u)$ [contain], or
2. $d(u) < f(u) < d(v) < f(v)$ [disjoint]

Proof of Parenthesis Theorem

- Consider the time when v is discovered
- Since u is discovered before v, there are two cases concerning the status of u :
  - Case 1: (u is not finished)
    This implies v is a descendant of u
    ➔ $f(v) < f(u)$ (why?)
  - Case 2: (u is finished)
    ➔ $f(u) < d(v)$

Corollary

Corollary:

v is a (proper) descendant of u

if and only if

$$d(u) < d(v) < f(v) < f(u)$$

Proof: v is a (proper) descendant of u

$\Leftrightarrow \quad d(u) < d(v)$ and $f(v) < f(u)$

$\Leftrightarrow \quad d(u) < d(v) < f(v) < f(u)$