CS5319 Advanced Discrete Structure

Homework 4
Due: December 12, 2013 (before class)

Quiz 2 will be held on December 17, 2013

1. Show that $n^4 - n^2$ is a multiple of 12 for all positive integer n.

2. Consider a round table with n position, where there is a piece of delicious cake at each position. Magnus and Derek are friends, and they want to play the following game. At the beginning, Magnus starts at a particular position (and takes the cake there). In each round Magnus chooses a positive integer m as the distance of the targeted seat from his current position, Derek decides a direction, clockwise or counterclockwise, for Magnus to move. The goal of Magnus is to visit as many different positions (and thus takes as many pieces of cake) as possible, while the goal of Derek is to counteract Magnus's goal, by minimizing the number of positions Magnus can move to.

   Show that when n is a power of 2, Magnus can always visit all the positions, no matter what Derek does.

3. Show that for any 51 integers selected from $[1, 100]$, we can always find two of them that are coprime (i.e., with no common factor greater than 1).

4. Given a rhombus (or a diamond) with length 2cm on each side, and the sharp angels are of 60 degrees. Prove that if we put 9 points inside the rhombus, there must be two points whose distance is at most 1 cm.

5. Consider a 3×9 grid with each grid point either colored red or white. Show that no matter how we color the points, there is always a rectangle in the grid whose four corners have the same color.

6. Consider a game played on an infinite checkerboard where each square below a certain horizontal line is occupied by a piece. Each move can jump a piece horizontally or vertically over another piece on to an empty square, where the jumped-over piece is then removed. Our target is to advance a piece as far away above the horizontal line as possible.

   For instance in Figure 1, we can make use of the red pieces to advance a piece two squares above the horizontal line (to the target square marked by a cross). Surprisingly, it was shown that there is a limit to which we can advance a piece.

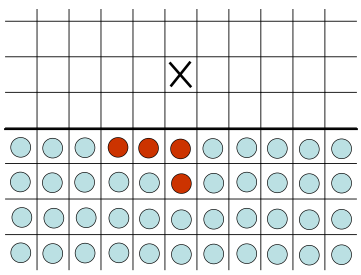

Figure 1: Advancing a piece above the horizontal line.

Suppose each square is associated with a value. In particular, the target square has a value 1, and each square with distance n from the target square has value x^n where $x = 0.9$. The *score* of the checkerboard is the total value of the squares occupied by all the pieces in the checkerboard.

(a) What is the score initially in terms of n?

(b) After a move, two pieces in the board will be replaced by a third one, so that there will be a change in the score. Show that when $x = 0.9$, the score will be decreased after each move.

(c) Based on the result of (b), give an upper bound on the value of n. (*Hint:* Recall that the target square has value 1.)

(d) By tuning the value of x, we can actually show $n < 5$, assuming the checkerboard has finite size. How?

7. (Challenging: No marks) Let A be a list of n integers between 1 and m. Let B be a list of m integers between 1 and n. Prove that there is a non-empty contiguous sublist of A and a non-empty contiguous sublist of B whose sums are the same.