CS 5319
Advanced Discrete Structure

Lecture 5:
Recurrence Relations I

Outline

- Introduction
- Linear Recurrence Relations with Constant Coefficients
- Solution by Generating Functions
- ★ Special Non-Linear Recurrences
- Recurrence with Two Indices

This Lecture

Introduction

Introduction

- Consider the sequence (1, 3, 9, 27, 81, …)
- We can describe the sequence by :

$$a_n = 3^n$$

- Alternatively, we can describe the sequence by expressing the nth term with the (n – 1)st term, together with the specification of the 1st term :

$$a_n = 3a_{n-1} \ , \quad a_0 = 1$$

Introduction

- For a sequence (a_0, a_1, a_2, …), an equation that relates a_n to some of its predecessor is called :

  recurrence relation or difference equation

- To start the computation, one must know several numbers in the sequence. These numbers are called the boundary conditions

- Ex :

$$a_n = 3a_{n-1} \quad \Leftrightarrow \quad \text{recurrence relation}$$

$$a_0 = 1 \quad \Leftrightarrow \quad \text{boundary condition}$$

Introduction

- Ex: Consider the Fibonacci sequence

(1, 1, 2, 3, 5, 8, 13, 21, …)

- The sequence can be described by the recurrence relation:

$$a_n = a_{n-1} + a_{n-2}$$

boundary conditions:

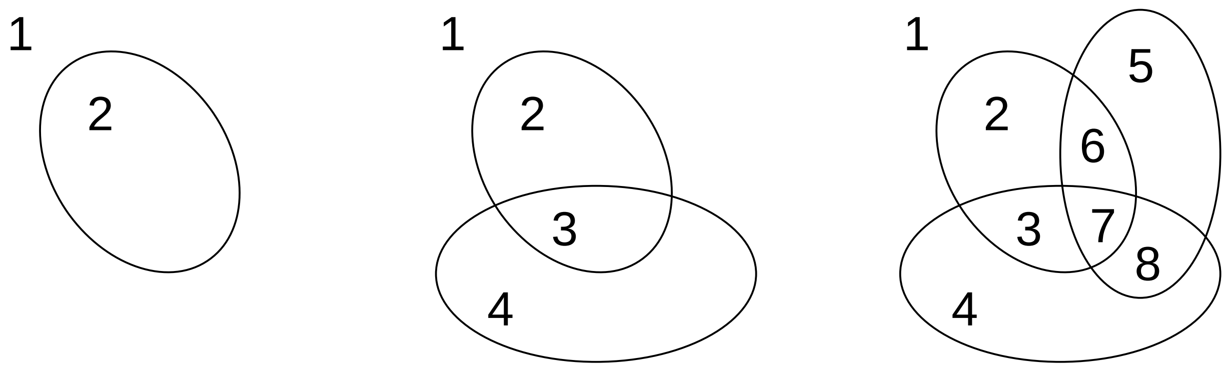
$$a_0 = 1 , \quad a_1 = 1$$

Introduction

- Ex: Let there be *n* ovals in the plane.

  Suppose that any two ovals intersect at exactly two points, and no three ovals intersect at the same point

Q: How many regions do they divide the plane ?



Introduction

- Let $a_n =$ number of regions divided by n ovals
- Suppose we have already drawn $n - 1$ ovals
- The nth oval will be divided into $2(n-1)$ arcs

➔ $$a_n = a_{n-1} + 2(n-1)$$

- Together with the boundary condition $a_1 = 2$, we can further compute

  $a_2 = 4$, $a_3 = 8$, $a_4 = 14$, $a_5 = 22$, $a_6 = 32$, …

Introduction

- In many cases obtaining the recurrence is a big step towards the solution to a counting problem
  - This is because even if we do not know the general expression, we can still compute the desired term a_n by a step-by-step approach

- Of course it is better if we can really obtain a general expression for a_n; in the following, we shall discuss a few ways of doing so

Linear Recurrence Relations with Constant Coefficients

Linear Recurrence Relations

- A recurrence relation of the form :

$$C_0 a_n + C_1 a_{n-1} + \ldots + C_r a_{n-r} = f(n)$$

is called a linear recurrence relation with constant coefficients, where all *C'*s are constants

- Ex : $3a_n - 5a_{n-1} + 2a_{n-2} = n^2 + 5$

Linear Recurrence Relations

- If r consecutive terms, say a_{k+1}, a_{k+2}, …, a_{k+r}, are known, then any a_n can be calculated recursively
- This implies that :

  Solution to the linear recurrence is determined uniquely by these r boundary conditions
- Indeed, we shall see that :

General form of the solution (when boundary conditions not fixed) has r unknown constants

Linear Recurrence Relations

- Let us begin with a simple example
- Suppose we know that

$$a_n + a_{n-1} = 2n + 1$$

- By observation, we may see that

$$a_n = n + 1$$

satisfies the above relation

Linear Recurrence Relations

- However, if we consider a similar recurrence

$$a_n + a_{n-1} = 0$$

we may notice that for any constant A

$$a_n = A \times (-1)^n$$

satisfies the above relation

Linear Recurrence Relations

- Consequently, we see that for any A,

$$a_n = n + 1 + A \times (-1)^n$$

will satisfy the original recurrence

$$a_n + a_{n-1} = 2n + 1$$

- If somehow we know the boundary condition, we can determine the value of A exactly
- Ex : If $a_0 = 8$, then $A = 7$

Linear Recurrence Relations

- Let us review what we have done
- In the above computation, we first find out a particular case where

$$a_n + a_{n-1} = 2n + 1$$

to obtain $a_n = n + 1$

- This is called a particular solution to the recurrence

Linear Recurrence Relations

- Next, we set the right side of the recurrence to 0.

$$a_n + a_{n-1} = 0$$

By solving this new recurrence, we obtain

$$a_n = A \times (-1)^n$$

- This is called the homogeneous solution to the recurrence

Linear Recurrence Relations

- The particular solution and the homogeneous solution together allow us to obtain a general (total) solution to the original recurrence :

$$a_n = n + 1 + A \times (-1)^n$$

- Finally, we resolve the unknowns based on the boundary conditions, and obtain the exact solution to the recurrence relation

Linear Recurrence Relations

- The particular solution is usually found by clever "guessing"
- Ex : To solve

$$a_n + 2a_{n-1} = n + 3$$

a reasonable guess is that

$$a_n = Bn + D$$

for some constants B and D

Linear Recurrence Relations

- Ex : To solve

$$a_n - a_{n-1} = n + 3$$

a reasonable guess is that

$$a_n = Bn^2 + Dn + E$$

for some constants B, D, and E

Linear Recurrence Relations

- Ex : To solve

$$a_n + 2a_{n-1} + a_{n-2} = 2^n$$

a reasonable guess is that

$$a_n = B * 2^n$$

for some constant B

Linear Recurrence Relations

- The homogeneous solution can be found by a systematic way
- To solve

$$C_0 a_n + C_1 a_{n-1} + \ldots + C_r a_{n-r} = 0$$

we first create a characteristic equation :

$$C_0 x^r + C_1 x^{r-1} + \ldots + C_r = 0$$

Linear Recurrence Relations

- Suppose α is a root to the characteristic equation. Then we see that for any constant A

$$a_n = A\alpha^n$$

is a homogeneous solution, because

$$C_0(A\alpha^n) + C_1(A\alpha^{n-1}) + \ldots + C_r(A\alpha^{n-r}) = 0$$

Linear Recurrence Relations

- Extending the previous arguments, if the characteristic equation has r distinct roots, denoted by $\alpha_1, \alpha_2, \ldots, \alpha_r$.
- Then we see that for any $A_1, A_2, \ldots, A_r$

$$a_n = A_1\alpha_1^n + A_2\alpha_2^n + \ldots + A_r\alpha_r^n$$

is a homogeneous solution

Linear Recurrence Relations

- Ex : The recurrence relation for Fibonacci sequence is :

$$a_n = a_{n-1} + a_{n-2}$$

This gives the characteristic equation :

$$x^2 - x - 1 = 0$$

Linear Recurrence Relations

- The two roots of the characteristic equation are :

$$\alpha = \frac{1+\sqrt{5}}{2} \quad , \quad \beta = \frac{1-\sqrt{5}}{2}$$

Thus the homogeneous solution is :

$$a_n = A\alpha^n + B\beta^n$$

This is also the total solution since the particular solution is 0

Linear Recurrence Relations

- Because $a_0 = 1$ and $a_1 = 1$, we have

$$1 = A + B \ , \quad 1 = A\alpha + B\beta$$

Thus, we can solve *A* and *B*, and finally obtain

$$a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$$

Linear Recurrence Relations

- Ex : Evaluate the $n \times n$ determinant

$$\begin{vmatrix}
1 & 1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & \ldots & 0 & 0 & 0 & 0 & 0 \\
.. & .. & .. & .. & .. & \ldots & .. & .. & .. & .. & .. \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 1 & 1
\end{vmatrix}$$

Linear Recurrence Relations

- Let a_n = the value of $n \times n$ determinant
- By expanding the determinant with respect to the first column, we get

$$a_n = a_{n-1} - \underbrace{\left.\begin{vmatrix} 1 & 0 & 0 & 0 & \ldots 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & \ldots 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & \ldots 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \ldots 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & \ldots 0 & 0 & 0 & 1 & 1 \end{vmatrix}\right\}}_{n-1} n-1$$

Linear Recurrence Relations

- Next by expanding the remaining determinant with respect to the first row, we get

$$a_n = a_{n-1} - a_{n-2}$$

This gives the characteristic equation :

$$x^2 - x + 1 = 0$$

Linear Recurrence Relations

- The two roots of the characteristic equation are :

$$\alpha = \frac{1 + \sqrt{3}\, i}{2}, \quad \beta = \frac{1 - \sqrt{3}\, i}{2}$$

Thus the homogeneous solution is :

$$a_n = A\alpha^n + B\beta^n$$

This is also the total solution since the particular solution is 0

Linear Recurrence Relations

- Since we are dealing with complex numbers, it is more convenient to express it in polar form :

$$\alpha = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\ ,\ \beta = \cos\frac{\pi}{3} - i\sin\frac{\pi}{3}$$

Thus the homogeneous solution becomes :

$$A\left(\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}\right) + B\left(\cos\frac{n\pi}{3} - i\sin\frac{n\pi}{3}\right)$$

Linear Recurrence Relations

- Because $a_1 = 1$ and $a_2 = 0$, we can solve A and B, and finally obtain

$$a_n = \frac{1}{2}\cos\frac{n\pi}{3} + \frac{1}{\sqrt{3}}\sin\frac{n\pi}{3}$$

Linear Recurrence Relations

- Observe: There is no i in the expression for a_n
  - Reason: All coefficients in the recurrence, and the boundary conditions, are real
- Thus we may at the beginning assume that

$$a_n = D\cos\frac{n\pi}{3} + E\sin\frac{n\pi}{3}$$

for some real numbers D and E, and solve for them

Linear Recurrence Relations

- So far, we have only considered the case where the characteristic equation has distinct roots.
- What if a root α is a double root ?
- We shall see that apart from $a_n = A\alpha^n$,

$$a_n = B\, n\, \alpha^n$$

is also a homogeneous solution

Linear Recurrence Relations

- First, we observe that α is a root to

$$C_0 n\, x^{n-1} + C_1 (n-1)\, x^{n-2} + \ldots + C_r (n-r)\, x^{n-r-1} = 0$$

so that

$$C_0 n\, \alpha^{n-1} + C_1 (n-1)\, \alpha^{n-2} + \ldots + C_r (n-r)\, \alpha^{n-r-1} = 0$$

- Consequently, this implies

$$a_n = B n \alpha^{n-1} \quad (\text{or } B n \alpha^n)$$

is a homogeneous solution

Linear Recurrence Relations

- Extending the previous arguments, if α is a k-multiple root of the characteristic equation, then
$$a_n = (A_1 n^k + A_2 n^{k-1} + \ldots + A_k)\, \alpha^n$$
is a homogeneous solution
- Further, if α_1, α_2, … are all roots of the equation such that α_j is a k_j-multiple root, then we have
$$k_1 + k_2 + \ldots = r\ ,$$
and the homogenous solution has r unknowns which can be found by boundary conditions

Linear Recurrence Relations

- Ex : To solve

$$a_n + 6a_{n-1} + 12a_{n-2} + 8a_{n-3} = 0$$

- The homogeneous solution is :

$$(A_1 n^2 + A_2 n + A_3)(-2)^n$$

so that we can solve for A_1, A_2, A_3 by boundary conditions

Linear Recurrence Relations

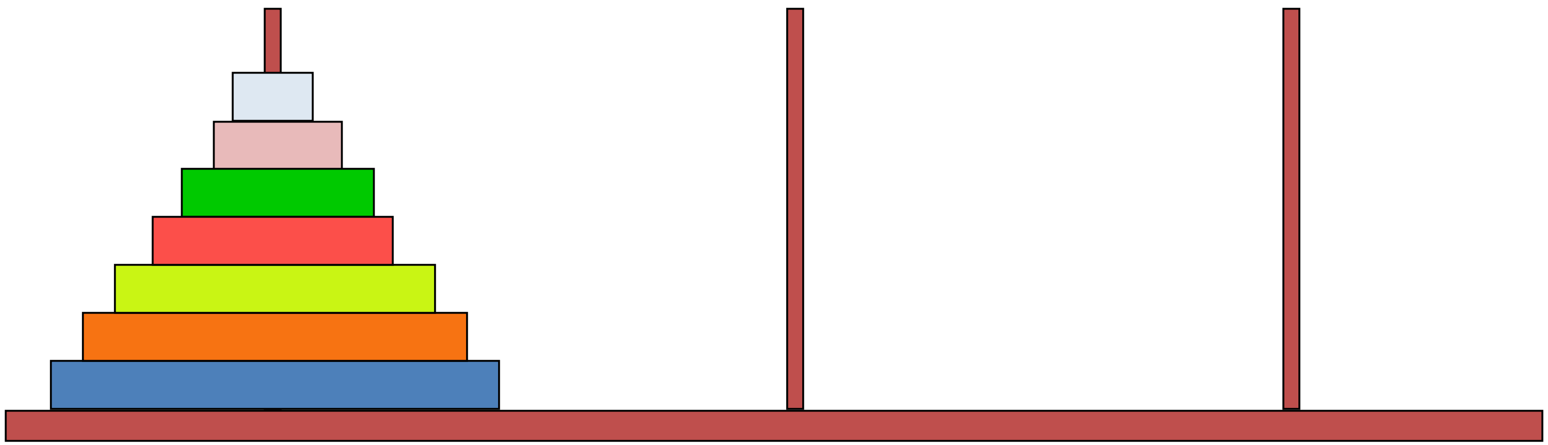
- Ex : Evaluate the $n \times n$ determinant

$$\begin{vmatrix}
2 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 2 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 2 & 1 & \dots & 0 & 0 & 0 & 0 & 0 \\
.. & .. & .. & .. & .. & \dots & .. & .. & .. & .. & .. \\
0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & 2
\end{vmatrix}$$

Linear Recurrence Relations

- Ex : The Tower of Hanoi

We are given a tower of *n* discs, initially stacked in decreasing size on one of the 3 pegs :

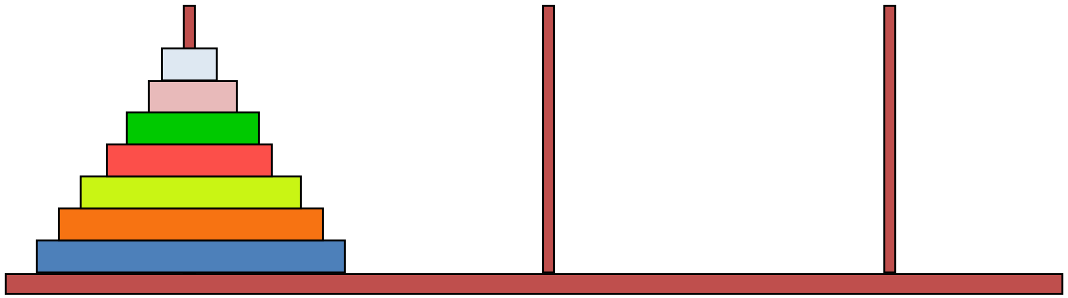


Linear Recurrence Relations

- Objective : Move entire tower to another peg
- Restrictions :

1. Move only one disc at a time, and
2. Never move a large one onto a smaller one

- Question : What is the minimum # of moves ?



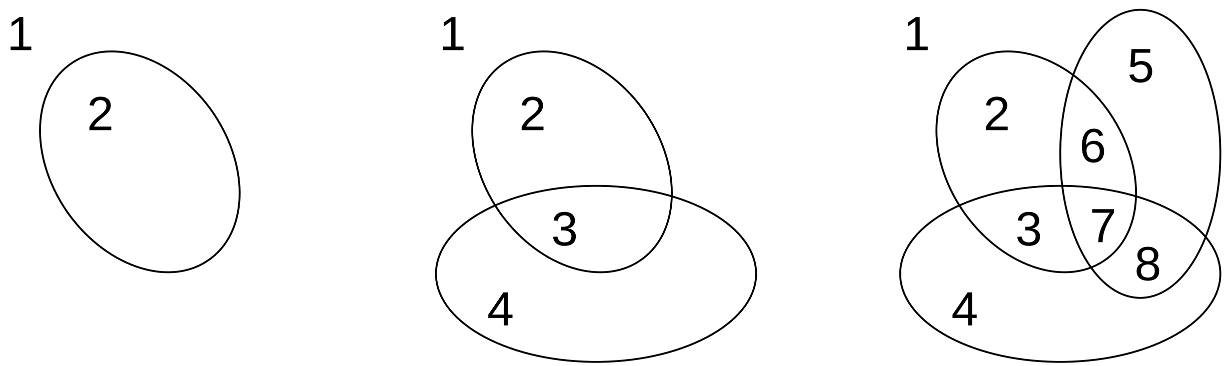
Solution by Generating Functions

Solution by GF

- Let us revisit an old problem

  Consider *n* ovals in the plane, where any two ovals intersect at exactly two points, and no three ovals intersect at the same point

Q: How many regions do they divide the plane ?



Solution by GF

- Let a_n = number of regions divided by n ovals
- Then we have $a_1 = 2$, and for any $n > 1$

$$a_n = a_{n-1} + 2(n-1)$$

- Now, let $A(x)$ be the GF of the values of a_n's :

$$A(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$$

and for consistency, we assume $a_0 = 2$

- Next, we use of $A(x)$ to obtain a formula for a_n

Solution by GF

- Firstly, we have

$$a_n x^n = a_{n-1} x^n + 2(n-1)\, x^n$$

which implies

$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} a_{n-1} x^n + \sum_{n=1}^{\infty} 2(n-1)\, x^n$$

- Thus we have :

$$A(x) - a_0 = x\,A(x) + 2x^2 / (1-x)^2$$

Solution by GF

- Next by rearranging terms, we get

$$A(x) = 2 / (1 - x) + 2x^2 / (1 - x)^3$$

- It follows that

$$\begin{aligned} a_n &= 2 + 2 \times C(-3, n-2)\,(-1)^{n-2} \\ &= 2 + 2 \times n\,(n-1) / 2 \\ &= 2 + n\,(n-1) \end{aligned}$$

Solution by GF

- In the previous derivation, we have set $a_0 = 2$ so that the formula

$$a_n = a_{n-1} + 2(n - 1)$$

can be applied even for $n = 1$

- Indeed, we can set a_0 to any arbitrary constant and the above method for solving a_n still work
  - But we will need some adjustments since the formula is no longer valid for $n = 1$

Solution by GF

- Let us set $a_0 = 5$ (an arbitrary constant)
- Then we have :

$$\sum_{n=2}^{\infty} a_n x^n = \sum_{n=2}^{\infty} a_{n-1} x^n + \sum_{n=2}^{\infty} 2(n-1)\, x^n$$

- Thus we have :

$$A(x) - a_1 x - a_0 = x\left(A(x) - a_0 \right) + 2x^2 / (1 - x)^2$$

Solution by GF

- Then by rearranging terms, we get

$$A(x) = 2x / (1 - x) + 2x^2 / (1 - x)^3 + 5$$

- It follows that

$$a_n = \begin{cases} 5 & n = 0 \\ n(n-1) + 2 & n = 1, 2, 3, \ldots \end{cases}$$

Solution by GF

- We next revisit another old problem

  Consider all n-digit quaternary strings.

  How many of them contains even # of 0's?

- Let a_n = # of n-digit strings with even 0's
- Previously, we have used combinatorial arguments or exponential GF to find a_n
- We now find a_n by first deriving a recurrence relation for a_n, and solve the relation using GF

Solution by GF

- Firstly, an n-digit string with even 0's can be obtained in one of the following ways :
  1. Obtain an $(n - 1)$-digit string with even 0's, and append either 1, 2, 3 at its end ;
  2. Obtain an $(n - 1)$-digit string with odd 0's, and append 0 at its end
- Thus for $n > 1$,

$$a_n = 3a_{n-1} + (4^{n-1} - a_{n-1})$$

Solution by GF

- After simplification, we get

$$a_n - 2a_{n-1} = 4^{n-1}$$

- Since $a_1 = 3$, we set $a_0 = 1$ so that the recurrence also holds for $n = 1$, and thus obtain :

$$\sum_{n=1}^{\infty} a_n x^n - \sum_{n=1}^{\infty} 2a_{n-1} x^n = \sum_{n=1}^{\infty} 4^{n-1} x^n$$

Solution by GF

- Consequently, we have

$$A(x) - 1 - 2x\,A(x) \;=\; x \,/\, (1 - 4x)$$

- By rearranging terms, we get

$$A(x) \;=\; \frac{1}{1-2x}\left(1 + \frac{x}{1-4x}\right)$$

$$= \frac{1/2}{1-2x} + \frac{1/2}{1-4x}$$

Solution by GF

- This immediately shows that

$$a_n = \frac{1}{2} 2^n + \frac{1}{2} 4^n$$

Solution by GF

- A challenging problem

  Consider all n-digit quaternary strings.

  How many of them contains even # of 0's and even # of 1's?

- Let b_n = # of n-strings with even 0's and even 1's

  c_n = # of n-strings with even 0's and odd 1's

  d_n = # of n-strings with odd 0's and even 1's

Solution by GF

- Thus for $n > 1$, we have :

$$b_n = 2b_{n-1} + c_{n-1} + d_{n-1}$$
$$c_n = b_{n-1} + 2c_{n-1} + 4^{n-1} - b_{n-1} - c_{n-1} - d_{n-1}$$
$$d_n = b_{n-1} + 2d_{n-1} + 4^{n-1} - b_{n-1} - c_{n-1} - d_{n-1}$$

- After simplification, we have :

$$b_n = 2b_{n-1} + c_{n-1} + d_{n-1}$$
$$c_n = c_{n-1} - d_{n-1} + 4^{n-1}$$
$$d_n = d_{n-1} - c_{n-1} + 4^{n-1}$$

Solution by GF

- Since we can choose b_0 , c_0, d_0 arbitrarily without affecting the result, we shall set

  $$b_0 = 3/4 , \quad c_0 = 1/4 , \quad d_0 = 1/4$$

  so that the previous recurrences is also valid for $n = 1$ (In fact, there are other sets of b_0, c_0, d_0)
- Next, we shall multiply both sides by x^n and obtain the sum for all $n \geq 1$
  - That is, we sum all the valid cases

Solution by GF

- Consequently, we obtain :

$$B(x) - \tfrac{3}{4} = 2x\,B(x) + x\,C(x) + x\,D(x)$$

$$C(x) - \tfrac{1}{4} = x\,C(x) - x\,D(x) + x\,/\,(1 - 4x)$$

$$D(x) - \tfrac{1}{4} = x\,D(x) - x\,C(x) + x\,/\,(1 - 4x)$$

- Solving the above, we get :

$$C(x) = D(x) = \tfrac{1}{4}\,/\,(1 - 4x)$$

$$B(x) = \tfrac{1}{4}\,/\,(1 - 4x) + \tfrac{1}{2}\,/\,(1 - 2x)$$

➔ $b_n = \tfrac{1}{4}\,4^n + \tfrac{1}{2}\,2^n$