CS 5319
Advanced Discrete Structure

Permutations and Combinations II

Outline

- Notation
- Rules of Sum and Product
- Permutations
- Combinations
- Distribution of Objects

* Stirling's Formula

This Lecture

Distribution of Objects

- A related topic is computing # ways to distribute objects to different positions

  Different treatments should be applied

  1. when objects are distinct or non-distinct
  2. when positions are distinct or non-distinct

- Next we study the case with *distinct positions*
  - non-distinct positions will be discussed later when we study generating functions

Distinct Objects

Distinct Objects

- Suppose we have r distinct objects and n cells
- Each cell can hold only 1 object

➔ When $n \geq r$, # of ways to put all objects into the cell is :

$$P(n,r)$$

➔ When $r \geq n$, # of ways to fill up the cells is :

$$P(r,n)$$

Distinct Objects

- Suppose we have *r* distinct objects and *n* cells
- Each cell can hold only *any number* of objects

➔ If we don't consider the order of objects within the same cell, # of ways is :

$$n^r$$

Distinct Objects

- Suppose we have r distinct objects and n cells
- Each cell can hold only *any number* of objects

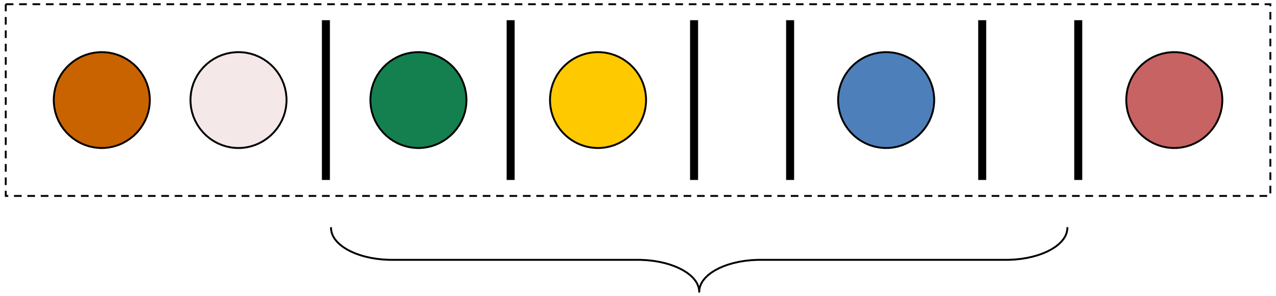
➔ If we also consider the order of objects within the same cell, # of ways is :

$$\frac{(n + r - 1)!}{(n - 1)!}$$

How to get this ??

Distinct Objects

- *Idea I:* Think of this as:



$n - 1$ bars to create n cells

- *Idea II:* Insert objects one by one. There are $n + j - 1$ positions for the j th object

Distinct Objects

Ex:

Suppose we have 7 flags and 5 masts.

All flags must be displayed, but not all masts have to be used.

How many ways can we arrange the flags on the masts ?

(Order of flags in the same mast is important)

Different Kinds of Objects

- Previously, we showed that for n objects with

  q_1 objects are of the first kind,

  q_2 objects are of the second kind,

  …

  q_t objects are of the t th kind.

➔ # of n-permutation of these objects is :

$$\frac{n!}{q_1!\, q_2! \ldots q_t!}$$

Different Kinds of Objects

- This can be viewed as putting these n objects into n cells, each cell can hold only one object
- To put the objects into the cells :

1. There are $C(n,q_1)$ ways to put objects of 1st kind
2. After that, there are $C(n - q_1, q_2)$ ways to put objects of 2nd kind
3. After that, there are $C(n - q_1 - q_2, q_3)$ ways to put objects of 3rd kind
4. And so on …

Different Kinds of Objects

- Thus # ways to distribute objects into the cells is :

$$C(n,q_1) \times C(n - q_1, q_2) \times C(n - q_1 - q_2, q_3) \times \ \ldots$$

$$\times\, C(n - q_1 - q_2 - \ldots - q_{t-1}\, , q_t)$$

$$= \quad \frac{n!}{q_1!\, q_2! \ldots q_t!}$$

Different Kinds of Objects

- Suppose we have only r objects ($n \geq r$) instead
- Each cell can still hold one object

➔ # ways to distribute the objects into the cells is :

$$C(n,q_1) \times C(n-q_1, q_2) \times C(n-q_1-q_2, q_3) \times \ \ldots$$
$$\times\, C(n-q_1-q_2-\ldots-q_{t-1}, q_t)$$

$$= \boxed{\frac{n!}{q_1!\, q_2!\, \ldots\, q_t!\, (n-r)!}}$$

Non-Distinct Objects

Non-Distinct Objects

- We have r non-distinct objects and n cells
- Each cell can hold only 1 object

➔ When $n \geq r$, # of ways to put all objects into the cell is :

$$C(n,r)$$

➔ When $r \geq n$, # of ways to fill up the cells = ??

Non-Distinct Objects

- We have r non-distinct objects and n cells
- Each cell can hold only *any number* of objects

→ # of ways to put objects into the cells is :

$$\frac{(n + r - 1)!}{(n - 1)!\, r!}$$

$$= C(n + r - 1, r)$$

Non-Distinct Objects

- We have r non-distinct objects and n cells
- Each cell holds at least one objects

(That is, we assume $r \geq n$)

➔ # of ways to put objects into the cells is :

$$C(r-1, n-1)$$

How about the case where each
cell holds at least q objects ??

Non-Distinct Objects

Ex:

Five distinct letters are to be transmitted through a communication channel.

A total of 15 blanks are to be inserted between the letters, with at least 3 blanks between every two letters.

How many ways can the letters and blanks be arranged ?

Non-Distinct Objects

Ex:

There are $2n + 1$ seats in a congress, to be divided among three parties.

In how many ways will some party obtain a majority of the seats ?

In how many ways will the coalition of any two parties ensure a majority of the seats ?

Stirling’s Formula

Stirling's Formula

- The values $n!$ or $\ln(n!)$ are frequently needed
- Exact calculation of these values are tedious
- In 1730, Stirling obtained an estimate for $n!$, which is now known as **Stirling's formula** or **Stirling's approximation** :

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

Stirling's Formula

- Although the absolute error is unbounded as *n* increases, that is :

$$\lim_{n\to\infty}\left[\, n! - \sqrt{2\pi n}\left(\frac{n}{e}\right)^{n} \right] \longrightarrow \infty$$

- The percentage error, however, is not :

$$\lim_{n\to\infty}\left[\, n! \Big/ \sqrt{2\pi n}\left(\frac{n}{e}\right)^{n} \right] \longrightarrow 1$$

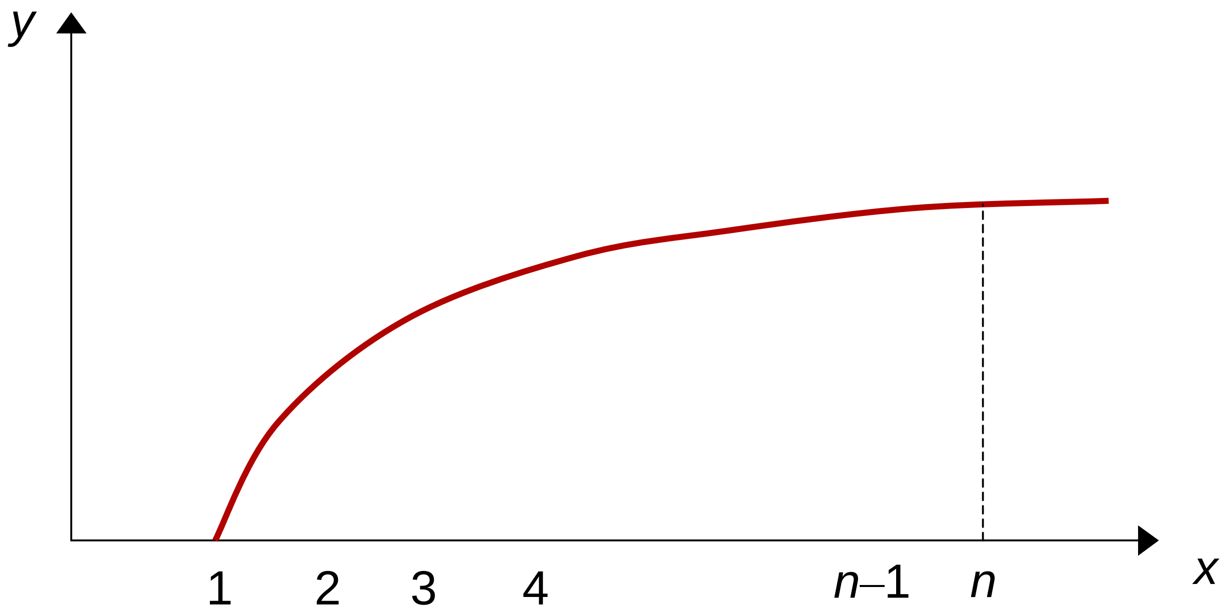
Stirling's Formula

Ex:

Value	Exact	Stirling	Error	% Error
1!	1	0.9221	0.08	8
2!	2	1.919	0.081	4
5!	120	118.019	1.981	1.7
100!	9.33×10^{157}	9.32×10^{157}	7.8×10^{154}	0.08

Stirling's Formula

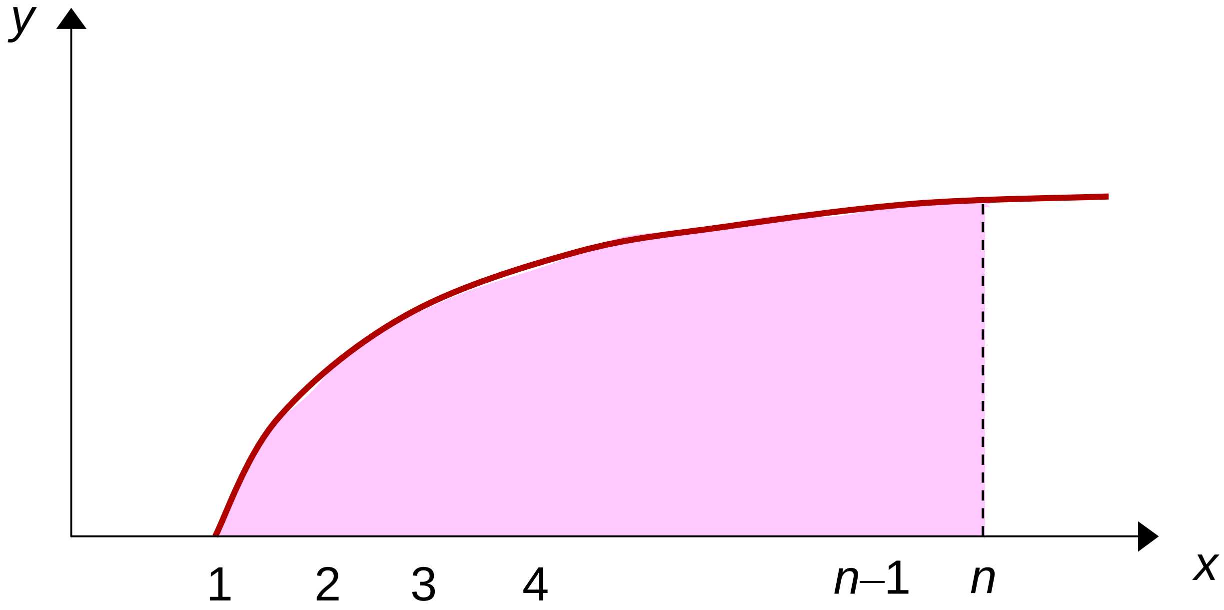
- To derive Stirling's formula, we will find a close estimate of $\ln(n!)$ using basic calculus
- Let us consider the curve $y = \ln x$



Stirling's Formula

- The area under the curve, from $x = 1$ to $x = n$, is equal to :

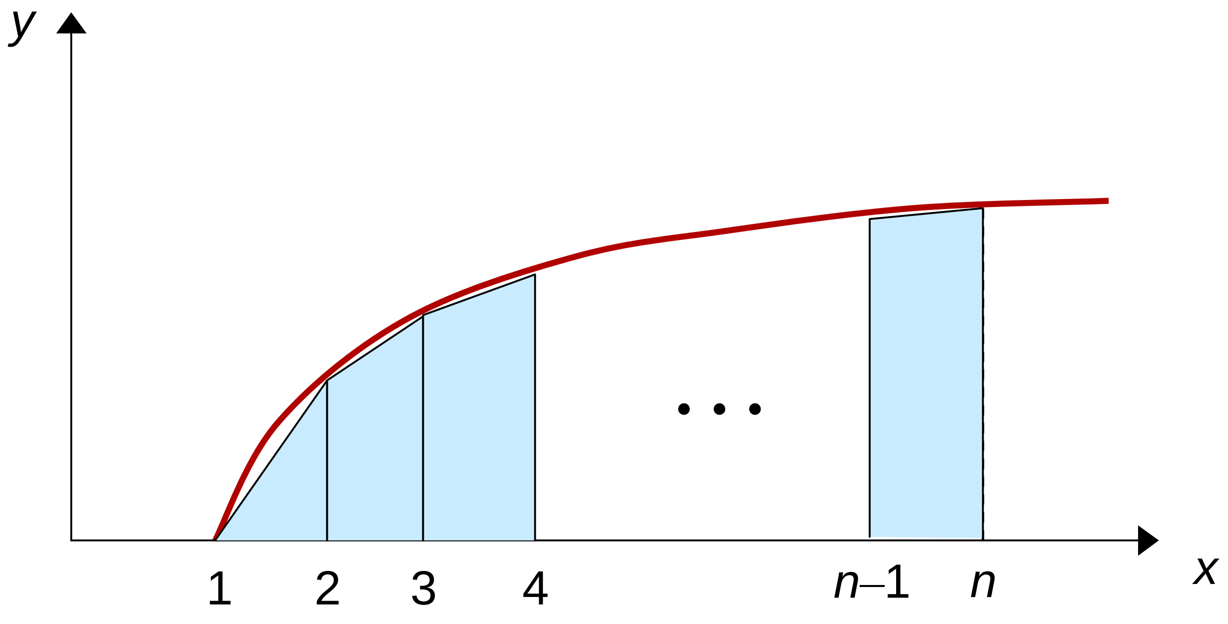
$$\int_1^n \ln x \, \mathrm{d}x$$



Stirling's Formula

- The area can be approximated by n trapeziums, so that

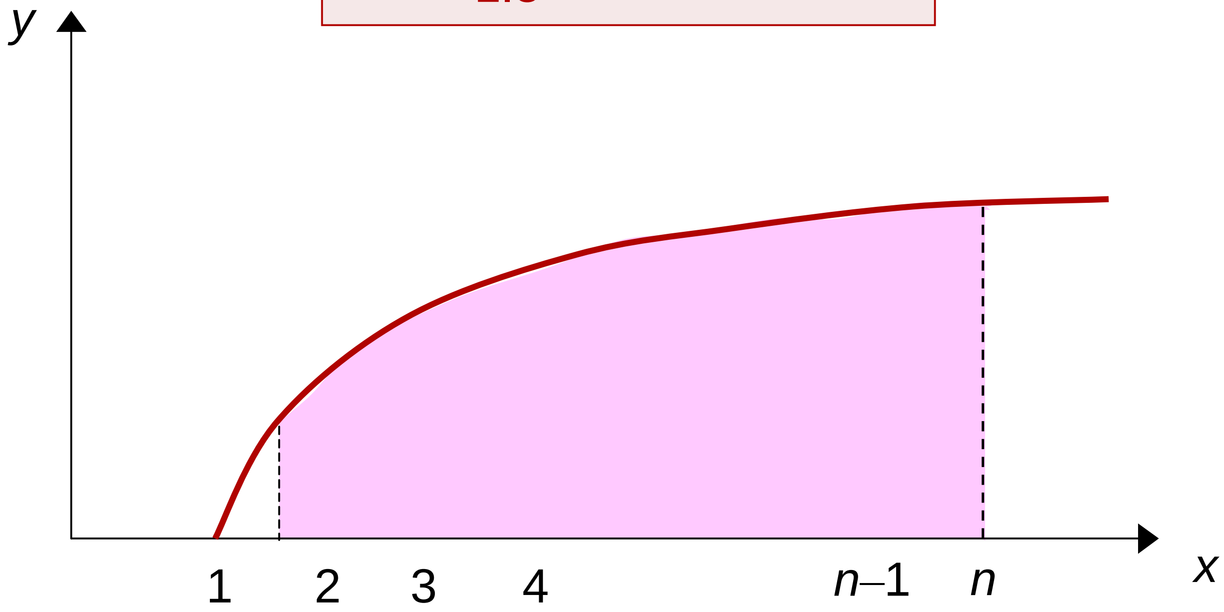
$$\int_1^n \ln x \, dx > \ln(n!) - 0.5 \ln n$$



Stirling's Formula

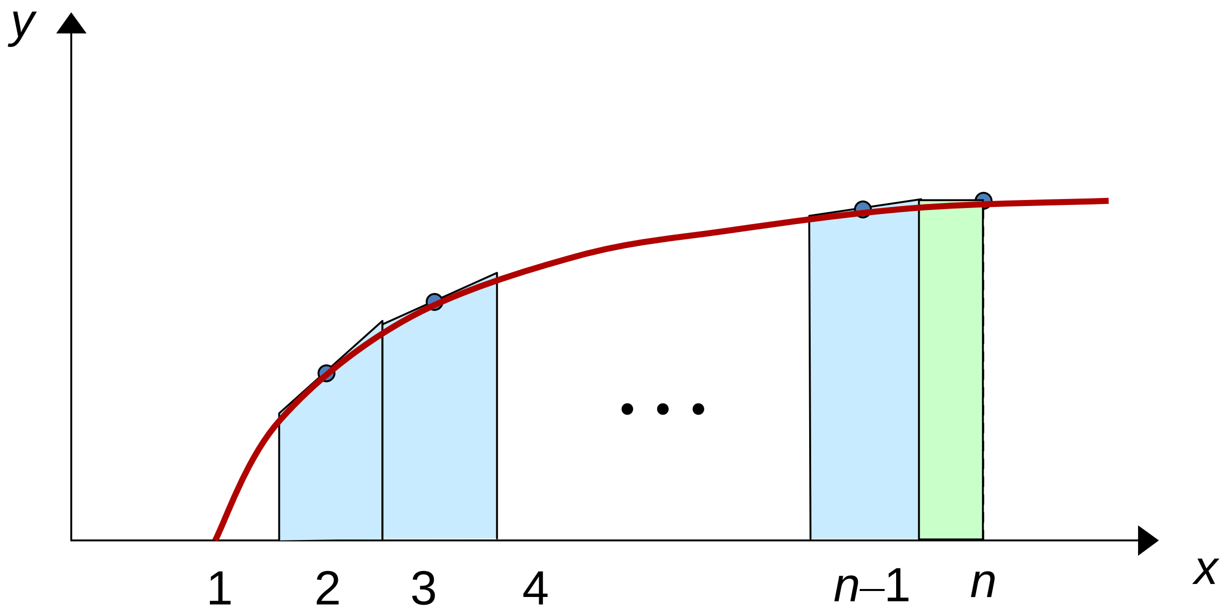
- On the other hand, the area under the curve, from $x = 1.5$ to $x = n$, is equal to :

$$\int_{1.5}^{n} \ln x \, dx$$



Stirling's Formula

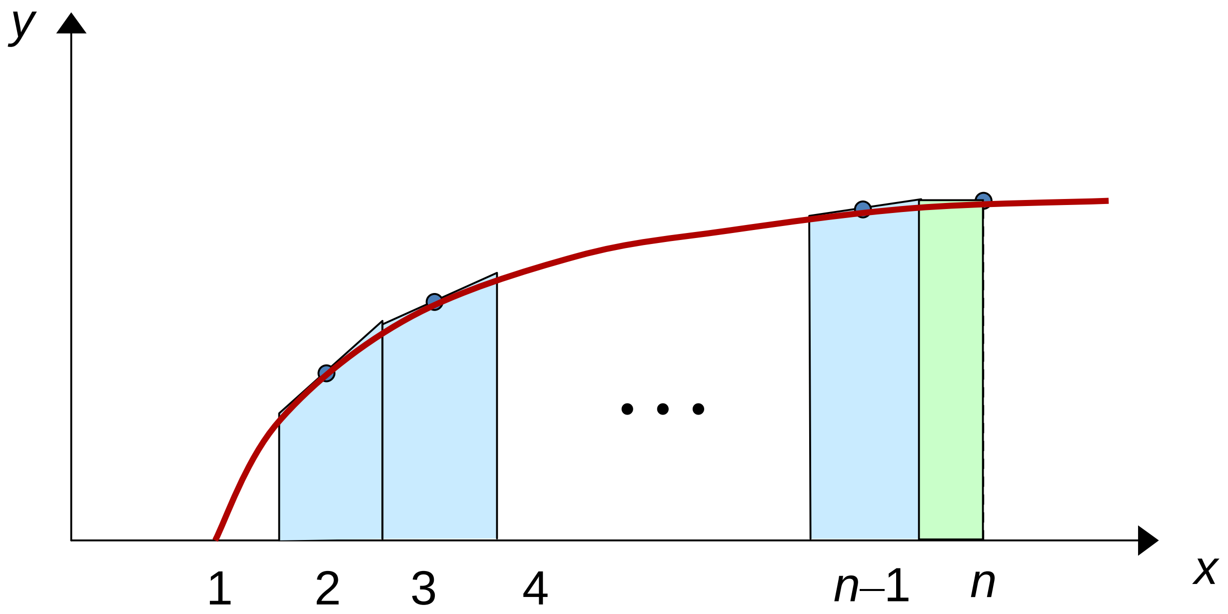
- Similarly, the latter area can be approximated by
  - $n - 1$ trapeziums (bounded by $x = k - 0.5$, $x = k + 0.5$, and tangent line at $(k, \ln k)$)
  - a last rectangle of size $0.5 \ln n$



Stirling's Formula

- Thus we have

$$\int_{1.5}^{n} \ln x \, dx \; < \; \ln(n!) - 0.5 \ln n$$



Stirling's Formula

- Combining the two inequalities, we have

$$\int_{1.5}^{n} \ln x \, dx < \ln(n!) - 0.5 \ln n < \int_{1}^{n} \ln x \, dx$$

- This implies

$$\ln(n!) = (n + 0.5) \ln n - n + \delta_n$$

where $1.5\,(1 - \ln 1.5) = 0.893 < \delta_n < 1$

Stirling's Formula

- By taking exponents on both sides,

$$n! = n^{(n + 0.5)} e^{-n} e^{\delta_n}$$

$$= e^{\delta_n} \sqrt{n} \left(\frac{n}{e}\right)^n$$

where $e^{0.893} < e^{\delta_n} < e$

- In fact, as n increases, e^{δ_n} will decrease monotonically, and approach the limit $\sqrt{2\pi}$