Advanced Discrete Structure Exam 1 Solution

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Question 1 (a)

Prove by combinatorial argument that the following identity is correct.

$$n \times C(n-1, r) \equiv (r+1) \times C(n, r+1)$$

Question 1 (a)

Suppose that we want to select $r+1$ people from n people to form a team, and one of them is the leader.

Question 1 (a)

(1) Select the leader first:

$$n \times C(n-1, r)$$

(2) Select the whole team first:

$$C(n, r+1) \times (r+1)$$

Therefore,

$$n \times C(n-1, r) \equiv (r+1) \times C(n, r+1)$$

Question 1 (b)

Prove the identity

$$C(n,1) + 2 \times C(n,2) + \cdots + n \times C(n,n)$$
$$= n \times 2^{n-1}.$$

Question 1 (b)

We know that

$$n \times C(n-1, r) \equiv (r+1) \times C(n, r+1)$$

Therefore

$$C(n,1) + 2 \times C(n,2) + \cdots + n \times C(n,n)$$

$$= \sum_{r=0}^{n-1} (r+1) \times C(n, r+1)$$

$$= n \times \sum_{r=0}^{n-1} C(n-1, r)$$

$$= n \times 2^{n-1}$$

Question 2

Find the number of n-digit strings generated from the alphabet $\{0, 1, 2\}$ whose total number of 0s and 1s is odd.

Question 2

The number of 0s is even and the number of 1s is odd:

$$\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)\times\left(x+\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots\right)$$

$$\times\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)$$

$$=\left(\frac{e^x+e^{-x}}{2}\right)\left(\frac{e^x-e^{-x}}{2}\right)\left(e^x\right)=\frac{e^{3x}-e^{-x}}{4}$$

Question 2

The number of 0s is odd and the number of 1s is even:

$$\left(x+\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots\right)\times\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)$$

$$\times\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)$$

$$=\left(\frac{e^x-e^{-x}}{2}\right)\left(\frac{e^x+e^{-x}}{2}\right)(e^x)=\frac{e^{3x}-e^{-x}}{4}$$

Question 2

The final GF:

$$\frac{e^{3x}-e^{-x}}{4}+\frac{e^{3x}-e^{-x}}{4}=\frac{e^{3x}-e^{-x}}{2}$$

The answer:

$$\frac{3^n-(-1)^n}{2}$$

Question 3

Let d_n be the number of ways to completely cover a $3 \times n$ rectangle with 3×1 dominoes. Find the generating function for $(d_0, d_1, d_2, \dots)$.

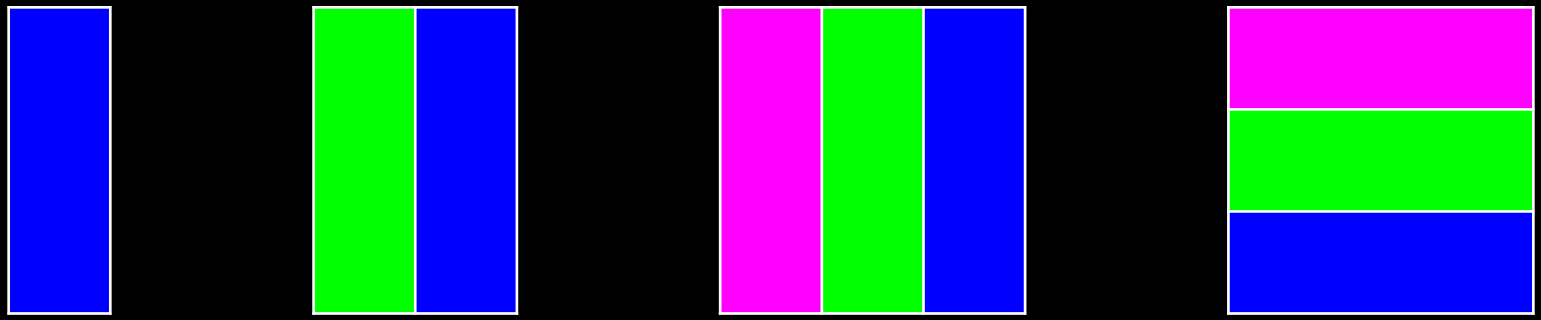
Question 3

$d_n = d_{n-1} + d_{n-3}$

$d_1 = d_2 = 1$

$d_3 = 2$

To satisfy the recurrence, let $d_0 = 1$.



Question 3

$$D(x) - d_2x^2 - d_1x - d_0 = x(D(x) - d_1x - d_0) + x^3D(x)$$
$$D(x) - x^2 - x - 1 = x(D(x) - x - 1) + x^3D(x)$$

The answer:

$$D(x) = \frac{1}{1 - x - x^3}$$

Question 4

Solve the recurrence

$$a_n = 3a_{n-1} - 4a_{n-3}$$

where $a_0 = 5$, $a_1 = 6$, and $a_2 = 22$.

Question 4

The particular solution: $a_n = 0$

The homogeneous solution:

$$x^3 = 3x^2 - 4$$

$$x^3 - 3x^2 + 4 = 0$$

$$(x+1)(x-2)^2 = 0$$

$$a_n = A(-1)^n + Bn2^n + C2^n$$

Question 4

$$a_n = A(-1)^n + Bn2^n + C2^n$$

$$a_0 = 5,\ a_1 = 6,\ a_2 = 22$$

$$\begin{cases} A + C = 5 \\ -A + 2B + 2C = 6 \\ A + 8B + 4C = 22 \end{cases}$$

$$(A, B, C) = (2,1,3)$$

The answer:

$$a_n = 2(-1)^n + (n+3)2^n$$

Question 5

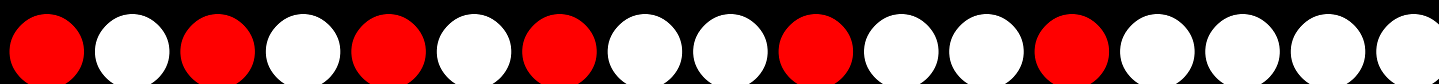
We want to choose a subset of six integers from $\{1,2,3,\ldots,17\}$ such that no consecutive integers are selected. E.g., $\{1,3,5,7,10,13\}$ can be chosen, but $\{2,3,5,7,10,13\}$ cannot. Find the exact value of the number of different subsets that can be chosen.

Question 5

Imagine that we have 17 balls.

Selected balls are red and others are white.

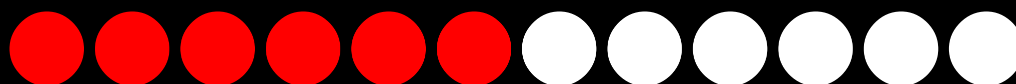
{1,3,5,7,10,13} :



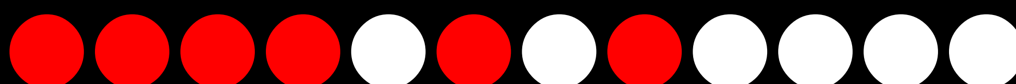
{2,3,5,7,10,13} :

Question 5

To ensure that there are no consecutive red balls, we can take out 5 white balls first:



Arrange the other balls:



Put the white balls back on the right of the first 5 red balls:

Question 5

Arrange the other balls:

$$\frac{12!}{6!\,6!}$$

Put back the white balls:

$$1$$

The answer:

$$\frac{12!}{6!\,6!} \times 1 = 924$$

Question 6 (a)

Show that the coefficient of x^{2k} in $(1 - x^2)^n$ is

$$(-1)^k \binom{n}{k}$$

Question 6 (a)

$$(1+a)^n = \sum_{i=0}^{n} \binom{n}{i} a^i$$

Let $a = -x^2$ and $i = k$:

$$\binom{n}{i} a^i = \binom{n}{k} (-x^2)^k = \binom{n}{k} (-1)^k x^{2k}$$

Question 6 (b)

Show that the coefficient of x^{m-2k} in $(1-x)^{-n}$ is

$$\binom{n+m-2k-1}{n-1}$$

Question 6 (b)

$$(1-x)^{-n} = (1+x+x^2+\cdots)^n$$

The coefficient of x^i is the number of ways to distribute *i* identical objects into *n* distinct groups.

The coefficient of x^{m-2k}:

$$\binom{(m-2k)+(n-1)}{n-1} = \binom{n+m-2k-1}{n-1}$$

Question 6 (c)

Evaluate the sum

$$\sum_{k=0}^{m/2} (-1)^k \binom{n}{k} \binom{n+m-2k-1}{n-1}$$

when $m \leq n$ and m is even.

Question 6 (c)

The coefficient of x^{2k} in $(1-x^2)^n$ is $(-1)^k\binom{n}{k}$.

The coefficient of x^{m-2k} in $(1-x)^{-n}$ is $\binom{n+m-2k-1}{n-1}$.

$\sum_{k=0}^{m/2}(-1)^k\binom{n}{k}\binom{n+m-2k-1}{n-1}$ should be the coefficient of x^m in $(1-x^2)^n \times (1-x)^{-n}$.

Question 6 (c)

$$(1-x^2)^n \times (1-x)^{-n} = \left(\frac{1-x^2}{1-x}\right)^n$$

$$= (1+x)^n$$

The coefficient of x^m:

$$\binom{n}{m}$$