Advanced Discrete Structure
Exam 1 Solution

Simon Chang
Prove by combinatorial argument that the following identity is correct.

\[ n \times C(n - 1, r) \equiv (r + 1) \times C(n, r + 1) \]
Question 1 (a)

Suppose that we want to select $r+1$ people from $n$ people to form a team, and one of them is the leader.
Question 1 (a)

(1) Select the leader first:
\[ n \times C(n - 1, r) \]

(2) Select the whole team first:
\[ C(n, r + 1) \times (r + 1) \]

Therefore,
\[ n \times C(n - 1, r) \equiv (r + 1) \times C(n, r + 1) \]
Question 1 (b)

Prove the identity

\[ C(n, 1) + 2 \times C(n, 2) + \cdots + n \times C(n, n) = n \times 2^{n-1}. \]
We know that
\[ n \times C(n - 1, r) \equiv (r + 1) \times C(n, r + 1) \]
Therefore
\[
C(n, 1) + 2 \times C(n, 2) + \cdots + n \times C(n, n) \\
= \sum_{r=0}^{n-1} (r + 1) \times C(n, r + 1) \\
= n \times \sum_{r=0}^{n-1} C(n - 1, r) \\
= n \times 2^{n-1}
\]
Question 2

Find the number of $n$-digit strings generated from the alphabet $\{0, 1, 2\}$ whose total number of 0s and 1s is odd.
The number of 0s is \textbf{even} and the number of 1s is \textbf{odd}:

\[
\left( 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \right) \times \left( x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \right) \times \left( 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \right)
\]

\[
= \left( \frac{e^x + e^{-x}}{2} \right) \left( \frac{e^x - e^{-x}}{2} \right) (e^x) = \frac{e^{3x} - e^{-x}}{4}
\]
The number of 0s is odd and the number of 1s is even:

\[
\left( x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \right) \times \left( 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \right) \\
\times \left( 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \right)
\]

\[
= \left( \frac{e^x - e^{-x}}{2} \right) \left( \frac{e^x + e^{-x}}{2} \right) (e^x) = \frac{e^{3x} - e^{-x}}{4}
\]
Question 2

The final GF:

\[
\frac{e^{3x} - e^{-x}}{4} + \frac{e^{3x} - e^{-x}}{4} = \frac{e^{3x} - e^{-x}}{2}
\]

The answer:

\[
\frac{3^n - (-1)^n}{2}
\]
Let $d_n$ be the number of ways to completely cover a $3 \times n$ rectangle with $3 \times 1$ dominoes. Find the generating function for $(d_0, d_1, d_2, \ldots)$. 
Question 3

\[ d_n = d_{n-1} + d_{n-3} \]
\[ d_1 = d_2 = 1 \]
\[ d_3 = 2 \]

To satisfy the recurrence, let \( d_0 = 1 \).
Question 3

\[ D(x) - d_2 x^2 - d_1 x - d_0 = x(D(x) - d_1 x - d_0) + x^3 D(x) \]
\[ D(x) - x^2 - x - 1 = x(D(x) - x - 1) + x^3 D(x) \]

The answer:

\[ D(x) = \frac{1}{1 - x - x^3} \]
Question 4

Solve the recurrence

\[ a_n = 3a_{n-1} - 4a_{n-3} \]

where \( a_0 = 5 \), \( a_1 = 6 \), and \( a_2 = 22 \).
The particular solution: \( a_n = 0 \)

The homogeneous solution:
\[
\begin{align*}
x^3 &= 3x^2 - 4 \\
x^3 - 3x^2 + 4 &= 0 \\
(x + 1)(x - 2)^2 &= 0 \\
a_n &= A(-1)^n + Bn2^n + C2^n
\end{align*}
\]
Question 4

\[ a_n = A(-1)^n + Bn2^n + C2^n \]

\[ a_0 = 5, \ a_1 = 6, \ a_2 = 22 \]

\[
\begin{aligned}
A + C &= 5 \\
-A + 2B + 2C &= 6 \\
A + 8B + 4C &= 22
\end{aligned}
\]

\[ (A, B, C) = (2, 1, 3) \]

The answer:

\[ a_n = 2(-1)^n + (n + 3)2^n \]
Question 5

We want to choose a subset of six integers from \{1,2,3, ..., 17\} such that no consecutive integers are selected. E.g., \{1,3,5,7,10,13\} can be chosen, but \{2,3,5,7,10,13\} cannot. Find the exact value of the number of different subsets that can be chosen.
Imagine that we have 17 balls. Selected balls are red and others are white.

\{1,3,5,7,10,13\}:

\{2,3,5,7,10,13\}:
To ensure that there are no consecutive red balls, we can take out 5 white balls first:

Arrange the other balls:

Put the white balls back on the right of the first 5 red balls:
Question 5

Arrange the other balls:
\[ \frac{12!}{6! \cdot 6!} \]

Put back the white balls:
\[ 1 \]

The answer:
\[ \frac{12!}{6! \cdot 6!} \times 1 = 924 \]
Question 6 (a)

Show that the coefficient of $x^{2k}$ in $(1 - x^2)^n$ is

$$(-1)^k \binom{n}{k}$$
(1 + a)^n = \sum_{i=0}^{n} \binom{n}{i} a^i

Let \( a = -x^2 \) and \( i = k \):

\[
\binom{n}{i} a^i = \binom{n}{k} (-x^2)^k = \binom{n}{k} (-1)^k x^{2k}
\]
Show that the coefficient of $x^{m-2k}$ in $(1 - x)^{-n}$ is

$$\binom{n + m - 2k - 1}{n - 1}$$
(1 − x)^{-n} = (1 + x + x^2 + \cdots)^n

The coefficient of \(x^i\) is the number of ways to distribute \(i\) identical objects into \(n\) distinct groups.

The coefficient of \(x^{m-2k}\):

\[
\binom{(m - 2k) + (n - 1)}{n - 1} = \binom{n + m - 2k - 1}{n - 1}
\]
Question 6 (c)

Evaluate the sum

$$\sum_{k=0}^{m/2} (-1)^k \binom{n}{k} \left( \begin{array}{c} n + m - 2k - 1 \\ n - 1 \end{array} \right)$$

when $m \leq n$ and $m$ is even.
The coefficient of $x^{2k}$ in $(1 - x^2)^n$ is $(-1)^k \binom{n}{k}$.

The coefficient of $x^{m-2k}$ in $(1 - x)^{-n}$ is $\binom{n+m-2k-1}{n-1}$.

$\sum_{k=0}^{m/2} (-1)^k \binom{n}{k} \binom{n+m-2k-1}{n-1}$ should be the coefficient of $x^m$ in $(1 - x^2)^n \times (1 - x)^{-n}$. 
Question 6 (c)

\[(1 - x^2)^n \times (1 - x)^{-n} = \left(\frac{1-x^2}{1-x}\right)^n\]

\[= (1 + x)^n\]

The coefficient of \(x^m\):

\[\binom{n}{m}\]