Advanced Discrete Structure Homework 6 Solution

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Question 1

Let $(A,*)$ be a semigroup.

Show that, for a, b, c in A, if $a * c = c * a$ and $b * c = c * b$, then $(a * b) * c = c * (a * b)$.

Question 1

Since it's a semigroup, the only property we can use is the associative property.

$$(a * b) * c = a * (b * c) = a * (c * b)$$
$$= (a * c) * b = (c * a) * b = c * (a * b)$$

Question 2

Let $(A,*)$ be a monoid such that for every x in A, $x * x = e$, where e is the identity element. Show that $(A,*)$ is a abelian group.

Question 2

We have to show that $a * b = b * a$ for any $a, b \in A$.

Again, we can only use the associative property.

Question 2

Since $b * a \in A$, $(b * a) * (b * a) = e$.

$$
\begin{aligned}
a * b &= (a * b) * (b * a) * (b * a) \\
&= (a * a) * (b * a) = b * a
\end{aligned}
$$

Question 3

Let $(H,\cdot)$ and $(K,\cdot)$ be subgroups of a group $(G,\cdot)$. Let

$$HK = \{h \cdot k \mid h \in H, k \in K\}$$

Show that $(HK,\cdot)$ is a subgroup if and only if $HK = KH$.

Question 3

$(HK,\cdot)$ is a subgroup $\rightarrow HK = KH$.

1. Show that for any member $x \in HK$, $x \in KH$.

2. Show that for any member $x \in KH$, $x \in HK$.

Question 3

1. Show that for any member $x \in HK$, $x \in KH$.

For any $x = h \cdot k$ in HK, $\exists h' \in H, \mathrm{k}' \in \mathrm{K}$ s.t.

$$(h \cdot k) \cdot (h' \cdot k') = e$$

Question 3

$$(h \cdot k) \cdot (h' \cdot k') \cdot (k')^{-1} \cdot (h')^{-1}$$
$$= (h \cdot k) \cdot h' \cdot (h')^{-1} = h \cdot k$$

$$(h \cdot k) \cdot (h' \cdot k') \cdot (k')^{-1} \cdot (h')^{-1}$$
$$= e \cdot (k')^{-1} \cdot (h')^{-1} = (k')^{-1} \cdot (h')^{-1}$$

$$h \cdot k = (k')^{-1} \cdot (h')^{-1} \in KH$$

Question 3

2. Show that for any member $x \in KH$, $x \in HK$.

$$(k \cdot h) \cdot (h^{-1} \cdot k^{-1}) = e$$

Since $(h^{-1} \cdot k^{-1}) \in HK$, $\exists (h' \cdot k') = (h^{-1} \cdot k^{-1})^{-1}$.

$$(k \cdot h) \cdot (h^{-1} \cdot k^{-1}) \cdot (h' \cdot k') = e \cdot h' \cdot k'$$
$$= h' \cdot k'$$
$$(k \cdot h) \cdot (h^{-1} \cdot k^{-1}) \cdot (h' \cdot k') = k \cdot h \cdot e$$
$$= k \cdot h$$
$$k \cdot h = h' \cdot k' \in HK$$

Question 3

$(HK,\cdot)$ is a subgroup $\leftarrow$ $HK = KH$.

1. Test whether $\cdot$ is a closed operation on HK.

2. Whether the identity element is in HK.

3. Whether each element in HK has an inverse.

Question 3

1. Test whether $\cdot$ is a closed operation on HK.

For any $h_1, h_2 \in H, k_1, k_2 \in K$,

$\exists h_3 \in H, k_3 \in K$ such that:

$$(h_1 \cdot k_1) \cdot (h_2 \cdot k_2) = h_1 \cdot (h_3 \cdot k_3) \cdot k_2$$

H and K are subgroups:

$$(h_1 \cdot h_3) \in H, (k_3 \cdot k_2) \in K$$

$\rightarrow (h_1 \cdot h_2) \cdot (k_1 \cdot k_2) \in HK$

Question 3

2. Whether the identity element is in HK.

H is a subgroup → H has a identity element e.

K is a subgroup → K has a identity element e.

$$e \cdot e \in HK$$

For any $h \in H, k \in K$:

$$(h \cdot k) \cdot (e \cdot e) = (e \cdot e) \cdot (h \cdot k) = h \cdot k$$

Question 3

3. Whether each element in HK has an inverse.

H is a subgroup → $\mathrm{h} \in H$ has an inverse h^{-1}.

K is a subgroup → $\mathrm{k} \in K$ has an inverse k^{-1}.

$$(h \cdot k) \cdot (k^{-1} \cdot h^{-1}) = (k^{-1} \cdot h^{-1}) \cdot (h \cdot k) = e$$

$$HK = KH$$

$$(k^{-1} \cdot h^{-1}) \in HK$$

Question 4

The *order* of an element a in a group is denoted to be the least positive integer m such that $a^m = e$, where e is the identity element. (If no positive power of a equals e, the order of a is denoted to be infinite.) Show that, in a finite group, the order of an element divides the order of the group.

Question 4

Let the order of $a \in (A,*)$ be m, $a^m = e$.

$(\{a, a^2, a^3, \cdots, a^m\},*)$ is a subgroup of $(A,*)$.

The size of $\{a, a^2, a^3, \cdots, a^m\}$ is m.

By Lagrange's Theorem, m divides $|A|$.

Question 5(a)

Determine the number of distinct 2×2 chessboards whose cells are painted white and black. Two chessboards are considered distinct if one cannot be obtained from another through rotation.

Question 5(a)

4 kinds of rotation:
(Cells with the same number must have the same color.)

Rotate 0°:

1	2
3	4

Rotate 90° left or right:

1	1
1	1

Question 5(a)

Rotate $180°$:

1	2
2	1

Use Burnside's Theorem:

$$\frac{2^4 + 2 + 2 + 2^2}{4} = 6$$

Question 5(b)

Repeat part (a) for 4×4 chessboards.

Question 5(b)

4 kinds of rotation:

Rotate 0°:

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

Rotate 90° left or right:

1	2	3	1
3	4	4	2
2	4	4	3
1	3	2	1

Question 5(b)

Rotate $180°$:

1	2	7	5
3	4	8	6
6	8	4	3
5	7	2	1

Use Burnside's Theorem:

$$\frac{2^{16} + 2^4 + 2^4 + 2^8}{4} = 16456$$

Question 6

Consider a cube with each face colored by one of the n colors. In how many distinct ways can the cube be colored?

(Two colorings are equal if one can be transformed to the other by rotating the cube.)

Question 6

Do nothing:

$$n^6$$

Holding 2 faces and rotate it by $90°$:

$$3 \times 2 \times n^3$$

Holding 2 faces and rotate it by $180°$:

$$3 \times n^4$$

Question 6

Holding 2 edges and rotate it by $180°$:

$$6 \times n^3$$

Holding 2 vertices and rotate it by $120°$:

$$4 \times 2 \times n^3$$

The answer:

$$\frac{n^6 + 3n^4 + 12n^3 + 8n^2}{24}$$