Advanced Discrete Structure
Homework 6 Solution

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Question 1

Let \((A,\ast)\) be a semigroup.

Show that, for \(a, b, c\) in \(A\), if \(a \ast c = c \ast a\) and \(b \ast c = c \ast b\), then \((a \ast b) \ast c = c \ast (a \ast b)\).
Question 1

Since it’s a semigroup, the only property we can use is the associative property.

\[(a * b) * c = a * (b * c) = a * (c * b)\]
\[= (a * c) * b = (c * a) * b = c * (a * b)\]
Let \((A,\ast)\) be a monoid such that for every \(x\) in \(A\), 
\[x \ast x = e,\]
where \(e\) is the identity element. Show that \((A,\ast)\) is a abelian group.
We have to show that $a \ast b = b \ast a$ for any $a, b \in A$.

Again, we can only use the associative property.
Since $b \ast a \in A$, $(b \ast a) \ast (b \ast a) = e$.

\[
a \ast b = (a \ast b) \ast (b \ast a) \ast (b \ast a)
\]
\[
= (a \ast a) \ast (b \ast a) = b \ast a
\]
Let \((H,\cdot)\) and \((K,\cdot)\) be subgroups of a group \((G,\cdot)\). Let

\[ HK = \{ h \cdot k \mid h \in H, k \in K \} \]

Show that \((HK,\cdot)\) is a subgroup if and only if \(HK = KH\).
Question 3

$(HK, \cdot)$ is a subgroup $\rightarrow HK = KH$.

1. Show that for any member $x \in HK$, $x \in KH$.
2. Show that for any member $x \in KH$, $x \in HK$. 
1. Show that for any member $x \in HK, x \in KH$.

For any $x = h \cdot k$ in $HK$, $\exists h' \in H, k' \in K$ s.t.

$$(h \cdot k) \cdot (h' \cdot k') = e$$
Question 3

\[(h \cdot k) \cdot (h' \cdot k') \cdot (k')^{-1} \cdot (h')^{-1} = (h \cdot k) \cdot h' \cdot (h')^{-1} = h \cdot k\]

\[(h \cdot k) \cdot (h' \cdot k') \cdot (k')^{-1} \cdot (h')^{-1} = e \cdot (k')^{-1} \cdot (h')^{-1} = (k')^{-1} \cdot (h')^{-1}\]

\[h \cdot k = (k')^{-1} \cdot (h')^{-1} \in KH\]
Question 3

2. Show that for any member $x \in KH$, $x \in HK$.

$$(k \cdot h) \cdot (h^{-1} \cdot k^{-1}) = e$$

Since $(h^{-1} \cdot k^{-1}) \in HK$, $\exists (h' \cdot k') = (h^{-1} \cdot k^{-1})^{-1}$.

$$(k \cdot h) \cdot (h^{-1} \cdot k^{-1}) \cdot (h' \cdot k') = e \cdot h' \cdot k'$$

$$= h' \cdot k'$$

$$(k \cdot h) \cdot (h^{-1} \cdot k^{-1}) \cdot (h' \cdot k') = k \cdot h \cdot e$$

$$= k \cdot h$$

$$k \cdot h = h' \cdot k' \in HK$$
Question 3

\((HK, \cdot)\) is a subgroup \(\iff HK = KH\).

1. Test whether \(\cdot\) is a closed operation on \(HK\).
2. Whether the identity element is in \(HK\).
3. Whether each element in \(HK\) has an inverse.
**Question 3**

1. Test whether \( \cdot \) is a closed operation on \( HK \).

For any \( h_1, h_2 \in H, k_1, k_2 \in K \),

\[ \exists h_3 \in H, k_3 \in K \text{ such that:} \]

\[ (h_1 \cdot k_1) \cdot (h_2 \cdot k_2) = h_1 \cdot (h_3 \cdot k_3) \cdot k_2 \]

\( H \) and \( K \) are subgroups:

\[ (h_1 \cdot h_3) \in H, (k_3 \cdot k_2) \in K \]

\[ \Rightarrow (h_1 \cdot h_2) \cdot (k_1 \cdot k_2) \in HK \]
Question 3

2. Whether the identity element is in $HK$.

$H$ is a subgroup $\rightarrow H$ has a identity element $e$.
$K$ is a subgroup $\rightarrow K$ has a identity element $e$.

$e \cdot e \in HK$

For any $h \in H, k \in K$:

$(h \cdot k) \cdot (e \cdot e) = (e \cdot e) \cdot (h \cdot k) = h \cdot k$
3. Whether each element in $HK$ has an inverse.

$H$ is a subgroup $\rightarrow h \in H$ has an inverse $h^{-1}$.

$K$ is a subgroup $\rightarrow k \in K$ has an inverse $k^{-1}$.

$(h \cdot k) \cdot (k^{-1} \cdot h^{-1}) = (k^{-1} \cdot h^{-1}) \cdot (h \cdot k) = e$

$HK = KH$

$(k^{-1} \cdot h^{-1}) \in HK$
The order of an element \( a \) in a group is denoted to be the least positive integer \( m \) such that \( a^m = e \), where \( e \) is the identity element. (If no positive power of \( a \) equals \( e \), the order of \( a \) is denoted to be infinite.) Show that, in a finite group, the order of an element divides the order of the group.
Question 4

Let the order of $a \in (A,\ast)$ be $m$, $a^m = e$. 
$\{a, a^2, a^3, \ldots, a^m\}$ is a subgroup of $(A,\ast)$.

The size of $\{a, a^2, a^3, \ldots, a^m\}$ is $m$.

By Lagrange’s Theorem, $m$ divides $|A|$. 
Question 5(a)

Determine the number of distinct $2 \times 2$ chessboards whose cells are painted white and black. Two chessboards are considered distinct if one cannot be obtained from another through rotation.
Question 5(a)

4 kinds of rotation:
(Cells with the same number must have the same color.)

Rotate 0°:

```
1 2
3 4
```

Rotate 90° left or right:

```
1 1
1 1
```
Question 5(a)

Rotate $180^\circ$:

\[
\begin{array}{cc}
1 & 2 \\
2 & 1 \\
\end{array}
\]

Use Burnside’s Theorem:

\[
\frac{2^4 + 2 + 2 + 2^2}{4} = 6
\]
Question 5(b)

Repeat part (a) for $4 \times 4$ chessboards.
Question 5(b)

4 kinds of rotation:

Rotate $0^\circ$:

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
</tr>
<tr>
<td>13</td>
<td>14</td>
<td>15</td>
<td>16</td>
</tr>
</tbody>
</table>

Rotate $90^\circ$ left or right:

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>4</td>
<td>4</td>
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<tr>
<td>2</td>
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<td>4</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>
Question 5(b)

Rotate 180°:

\[
\begin{array}{cccc}
1 & 2 & 7 & 5 \\
3 & 4 & 8 & 6 \\
6 & 8 & 4 & 3 \\
5 & 7 & 2 & 1 \\
\end{array}
\]

Use Burnside’s Theorem:

\[
\frac{2^{16} + 2^4 + 2^4 + 2^8}{4} = 16456
\]
Consider a cube with each face colored by one of the $n$ colors. In how many distinct ways can the cube be colored?

(Two colorings are equal if one can be transformed to the other by rotating the cube.)
Question 6

Do nothing:

\[ n^6 \]

Holding 2 faces and rotate it by 90°:

\[ 3 \times 2 \times n^3 \]

Holding 2 faces and rotate it by 180°:

\[ 3 \times n^4 \]
Question 6

Holding 2 edges and rotate it by 180°:
\[ 6 \times n^3 \]

Holding 2 vertices and rotate it by 120°:
\[ 4 \times 2 \times n^3 \]

The answer:
\[
\frac{n^6 + 3n^4 + 12n^3 + 8n^2}{24}
\]