# CS5319 Advanced Discrete Structure

Homework 6

Due: 3:20 pm, December 29, 2011 (before class)

1. Let $(A, *)$ be a semigroup. Show that, for $a$, $b$, $c$ in $A$, if $a * c = c * a$ and $b * c = c * b$, then $(a * b) * c = c * (a * b)$.

2. Let $(A, *)$ be a monoid such that for every $x$ in $A$, $x * x = e$, where $e$ is the identity element. Show that $(A, *)$ is a abelian group.

3. Let $(H, \cdot)$ and $(K, \cdot)$ be subgroups of a group $(G, \cdot)$. Let

$$HK = \{h \cdot k \mid h \in H, k \in K\}.$$

   Show that $(HK, \cdot)$ is a subgroup if and only if $HK = KH$.

4. The *order* of an element $a$ in a group is defined to be the least positive integer $m$ such that $a^m = e$, where $e$ is the identity element. (If no positive power of $a$ equals $e$, the order of $a$ is defined to be infinite.) Show that, in a finite group, the order of an element divides the order of the group.

5. (a) Determine the number of distinct $2 \times 2$ chessboards whose cells are painted white and black. Two chessboards are considered distinct if one cannot be obtained from another through rotation.

   (b) Repeat part (a) for $4 \times 4$ chessboards.

6. Consider a cube with each face colored by one of the $n$ colors. In how many distinct ways can the cube be colored? (Two colorings are equal if one can be transformed to the other by rotating the cube.)