Advanced Discrete Structure Homework 4 Solution

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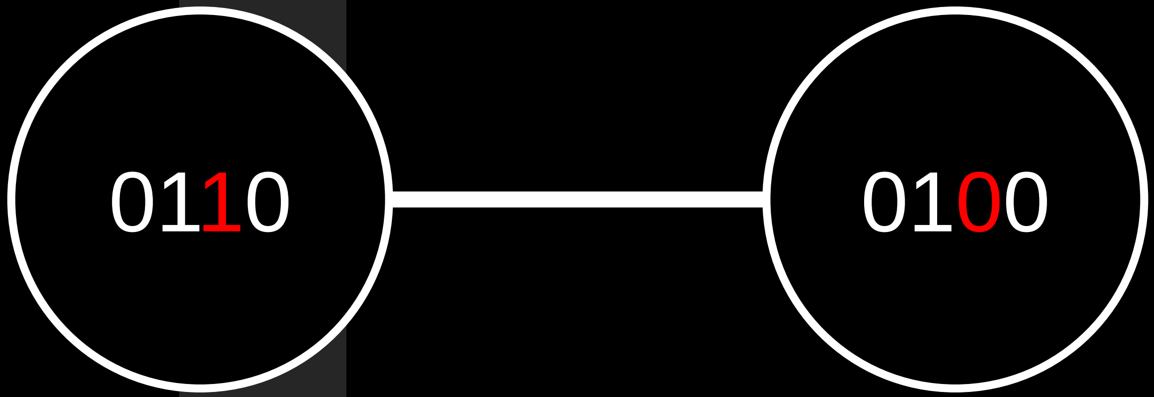
1

Show that for any k, there is a graph with 2^k vertices such that the graph is k-regular.

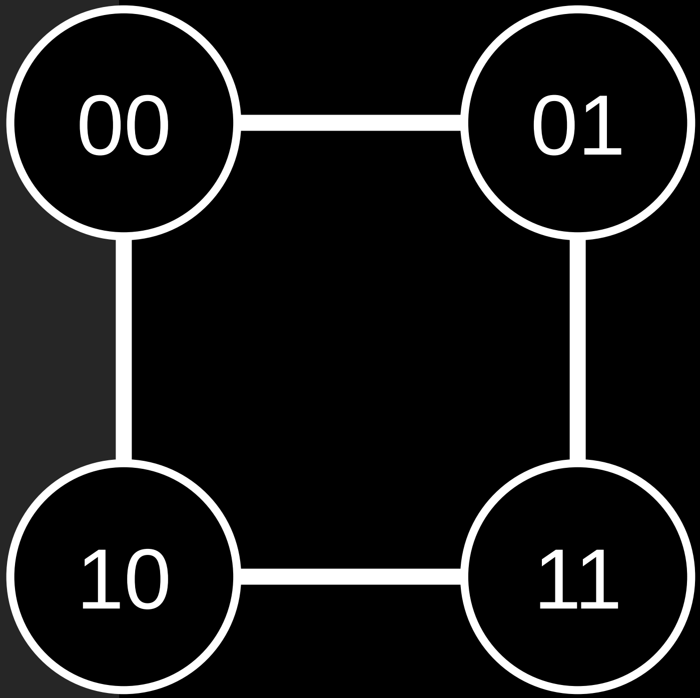
1. Let the vertices represent different k-bit binary strings.

2. Connects the vertices with only one different bit.

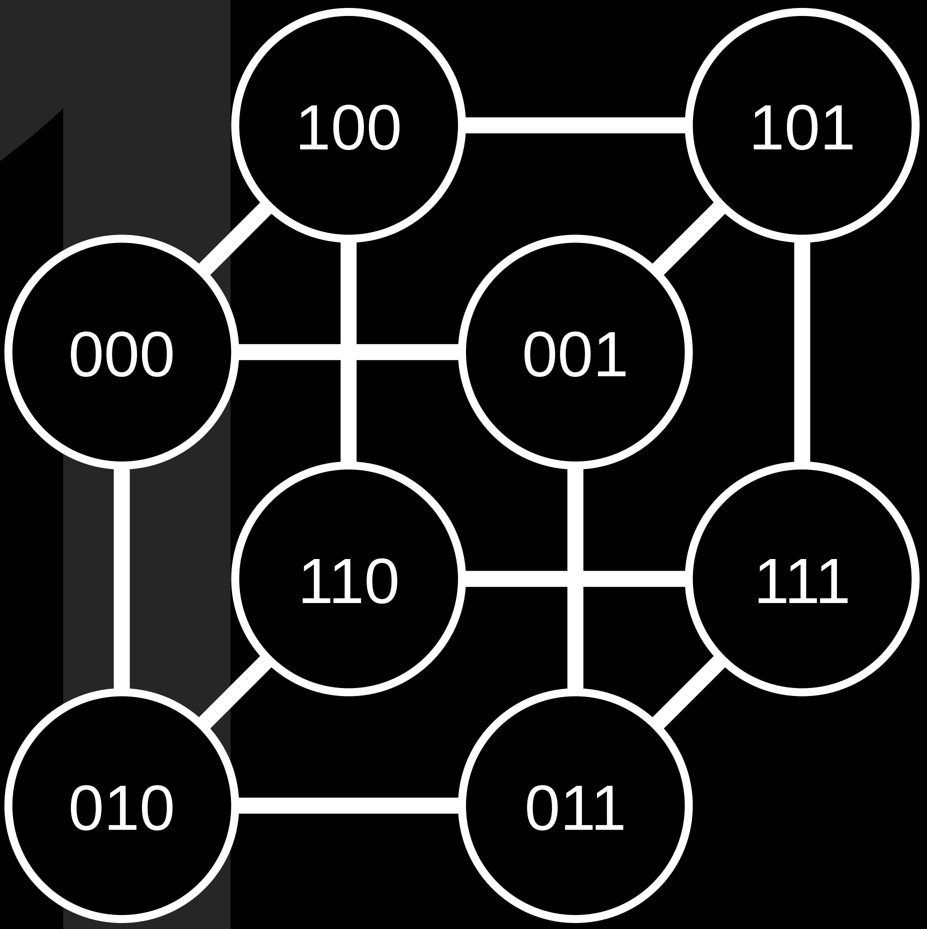
→ Every vertex has exactly k neighbors.



For k = 2:



For k = 3:



2

$$F_n = 2^{2^n} + 1$$

2(a)

Show that for all $n \geq 1$

$$F_n = F_0 \times F_1 \times \cdots \times F_{n-1} + 2.$$

Prove it by induction:

Base step: *n = 1*

$$F_1 = F_0 + 2 = \left(2^{2^0} + 1\right) + 2 = 5 = 2^{2^1} + 1$$

Induction step:

Suppose that $F_k = F_0 \times F_1 \times \cdots \times F_{k-1} + 2$ is true.

$$\begin{aligned} F_{k+1} &= F_0 \times F_1 \times \cdots \times F_{k-1} \times F_k + 2 \\ &= (F_k - 2) \times F_k + 2 \\ &= \left(2^{2^k} - 1\right)\left(2^{2^k} + 1\right) + 2 = 2^{2^{k+1}} + 1 \end{aligned}$$

Done!

2(b)

Using the result of (a), argue that Fermat numbers are pairwise relatively prime.

Assume $m < n$:

$$F_n = F_0 \times F_1 \times \cdots \times F_m \times \cdots \times F_{n-1} + 2$$

or

$$F_n = F_0 \times F_1 \times \cdots \times F_m + 2$$

Use Euclidean algorithm:

$$\gcd(F_m, F_n) = \gcd(F_m, 2) = \gcd\left(2^{2^m} + 1, 2\right)$$
$$= \gcd(1,2) = 1$$

2(b)

Show that if we pick one prime factor from each Fermat number, they must be all distinct.

Ans: If they are not distinct, there will be Fermat numbers that are not pairwise relatively prime.

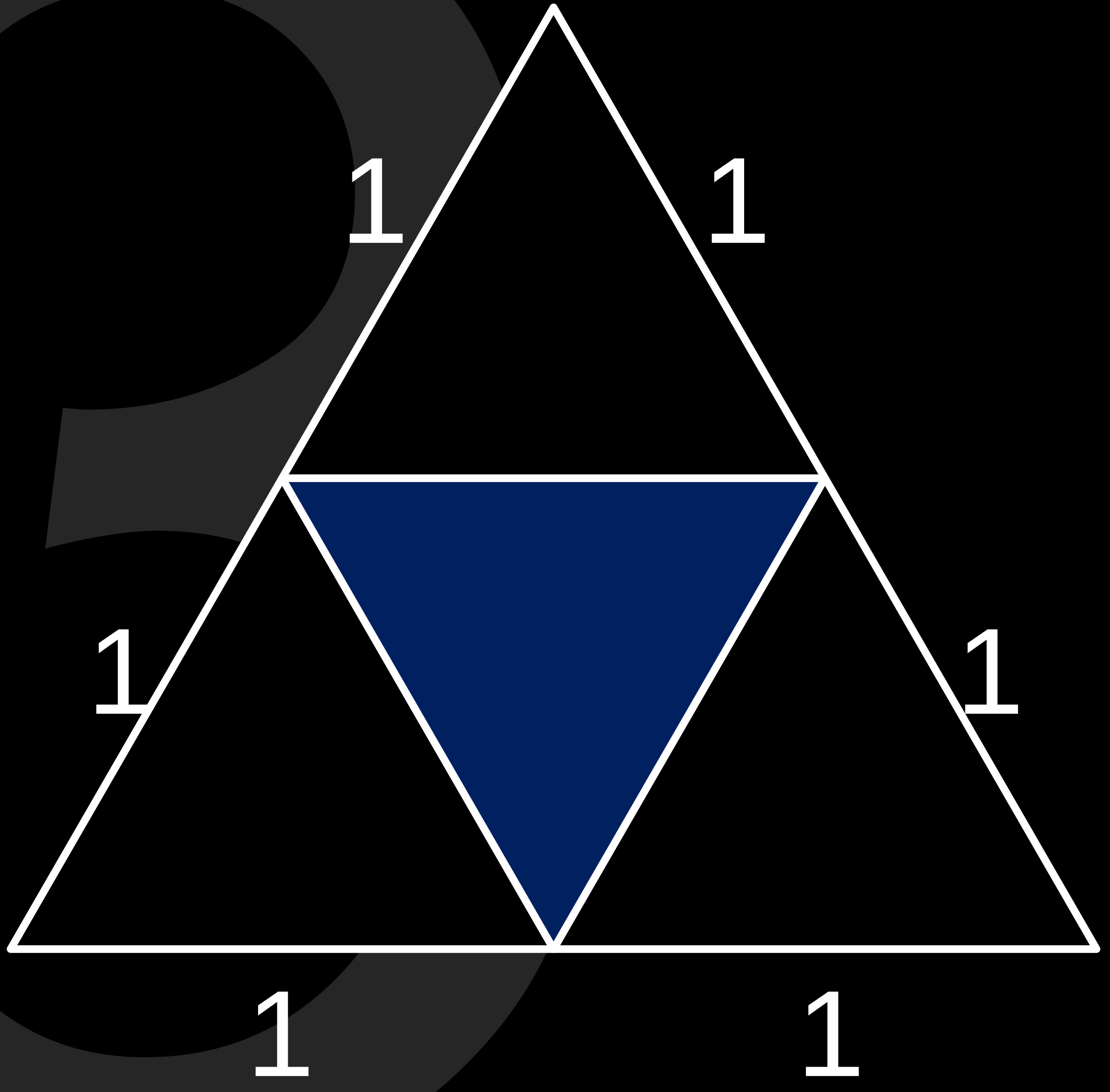
2(d)

Using the result of (c), conclude that there are infinitely many primes.

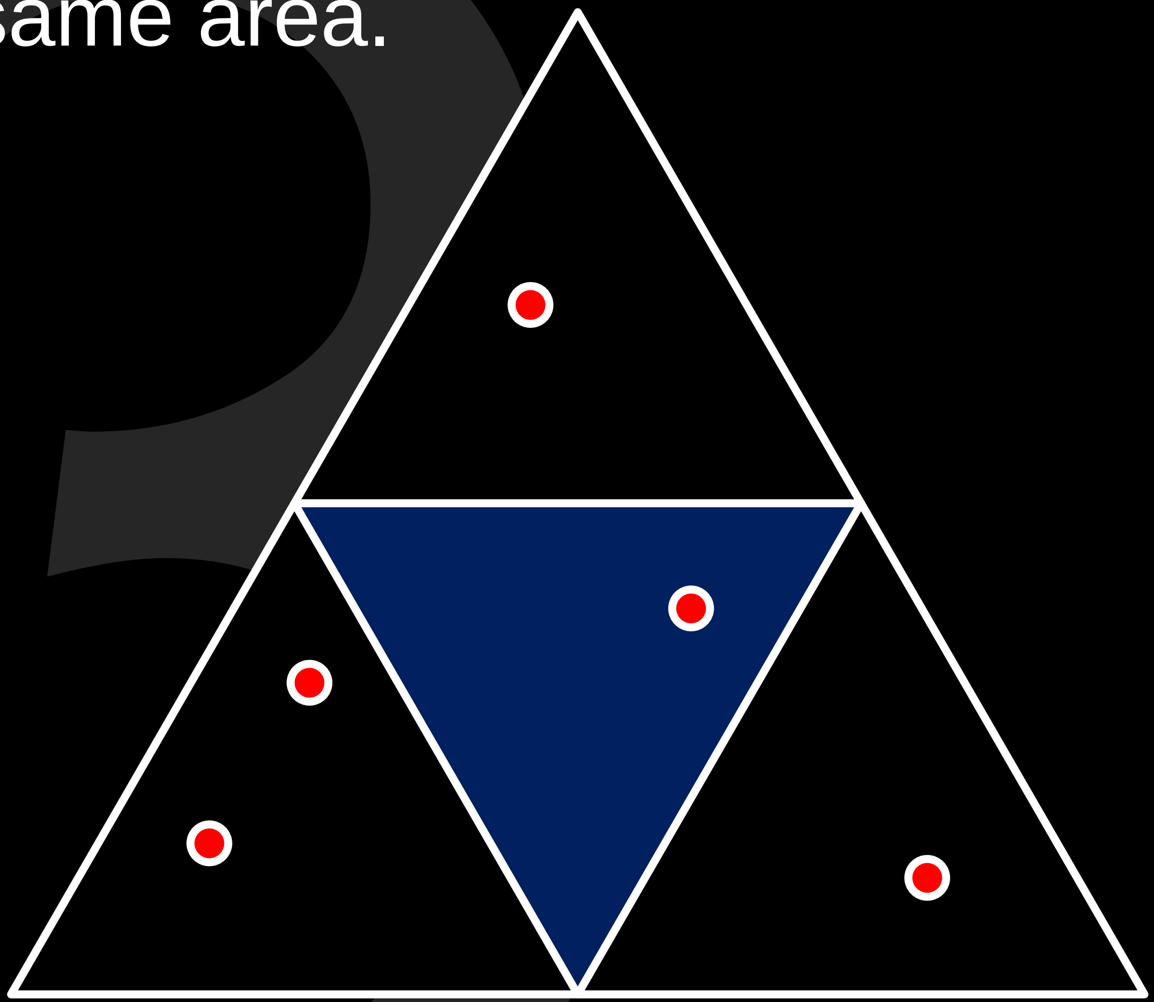
Ans: We can pick one distinct prime from each Fermat number, and there are infinitely many Fermat numbers.

Let Δ denote an equilateral triangle with the length of each side equal to *2* units. Show that by placing 5 points inside Δ, we can always find two points whose distance is at most *1* unit.

We can divide the triangle into *4* parts:



Put 5 points in the triangle, there will be two in the same area.



4

Show that among $n + 2$ arbitrarily chosen integers, either there are two whose difference is divisible by $2n$ or there are two whose sum is divisible by $2n$.

The integers are classified into these groups:

$$k(2n) \pm 0, k(2n) \pm 1, k(2n) \pm 2, \ldots, k(2n) \pm n$$

The sum or difference of the integers belong to the same group is divisible by $2n$.

There are n + 1 groups and n + 2 integers.
→ There are two integers in the same group.

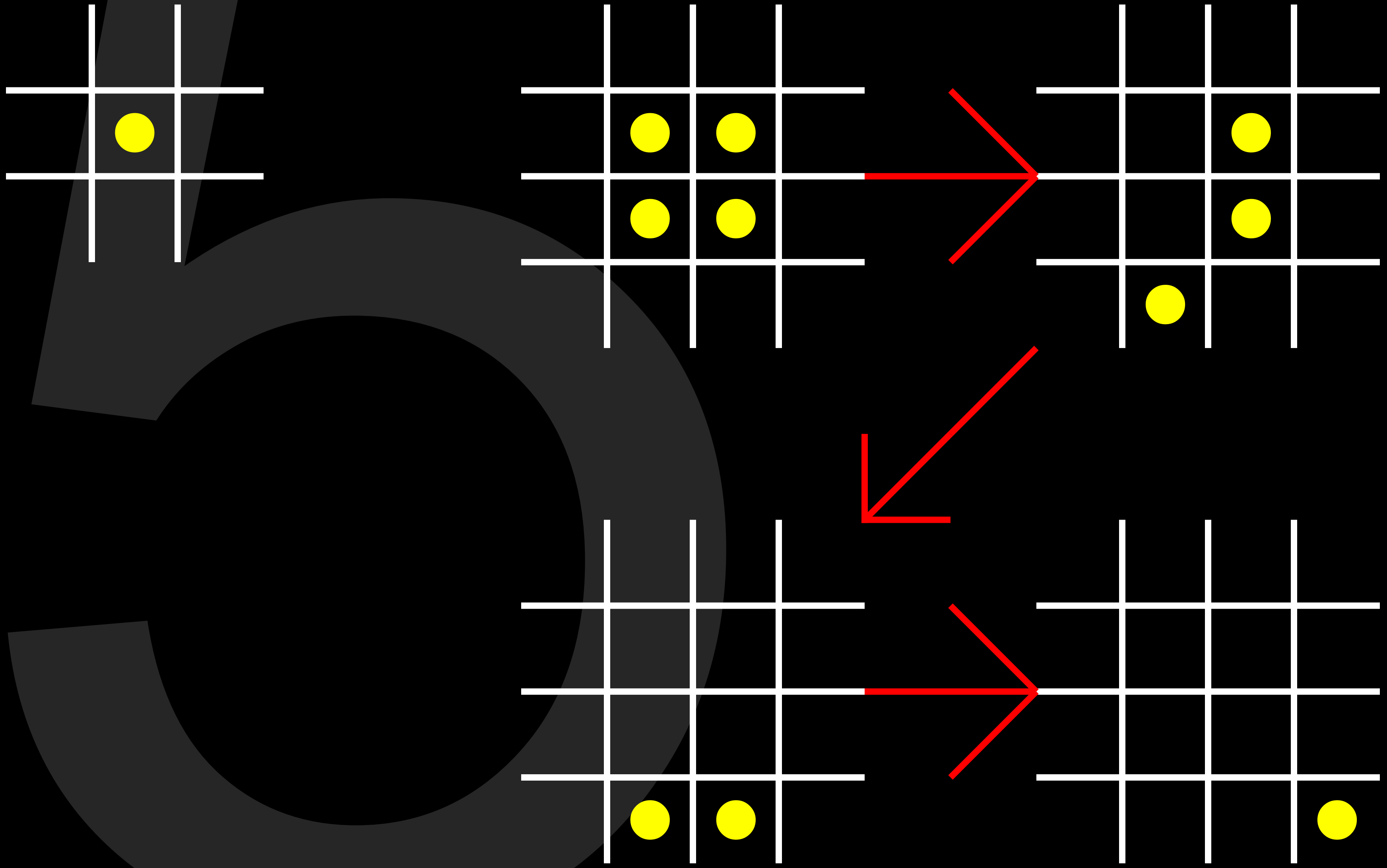
Done!

5

Consider a game played on an infinite checkerboard where there is an $n \times n$ space and each square in it is occupied by a piece. Each move can jump a piece horizontally or vertically over another piece on to an empty square, where the jumped-over piece is then removed. The target is to remove the pieces so that there is only one left. Prove that it is possible when n is not a multiple of 3.

Prove by induction:

Base case: $n = 1$ or $n = 2$

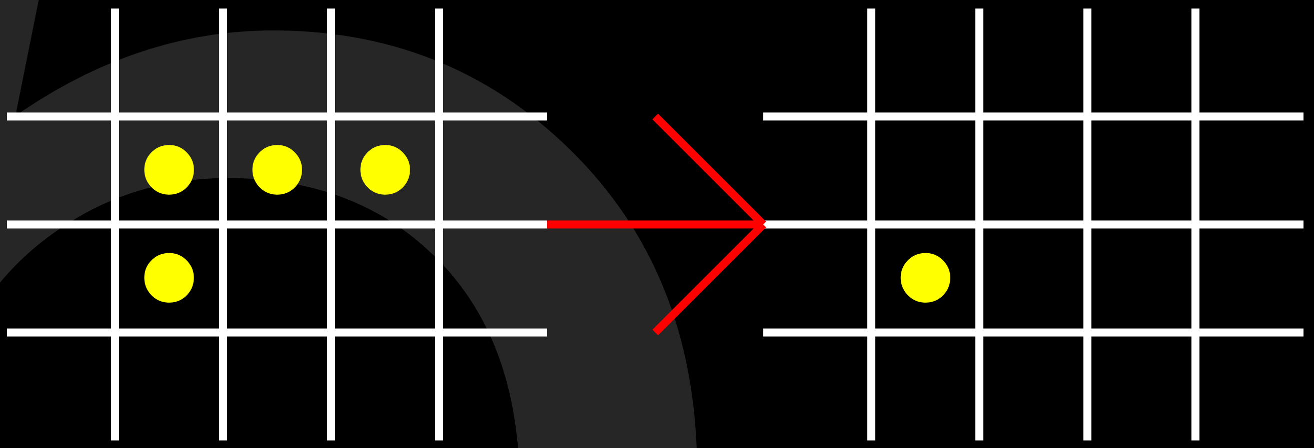


Induction step:

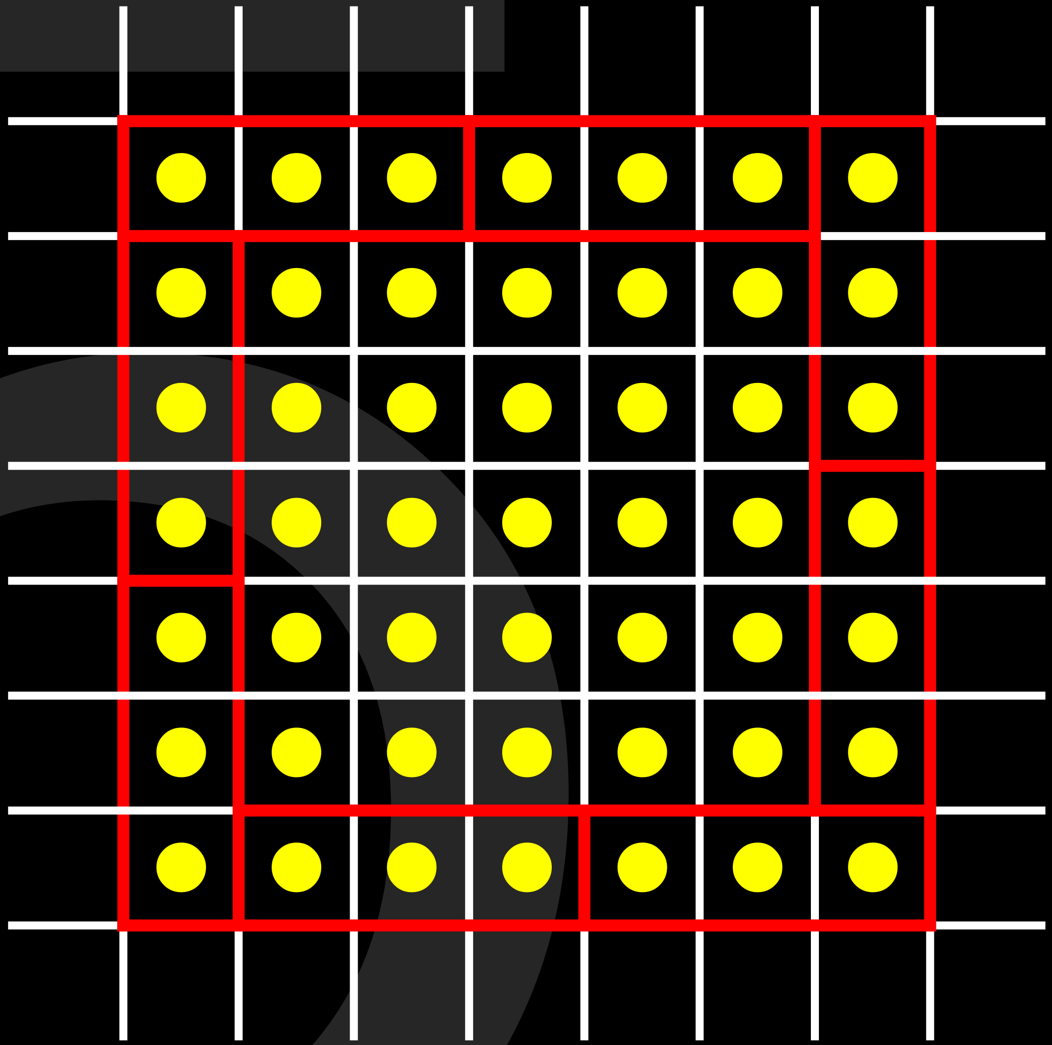
Assume it can be solved when

$n = 3k + 1$ or $n = 3k + 2$

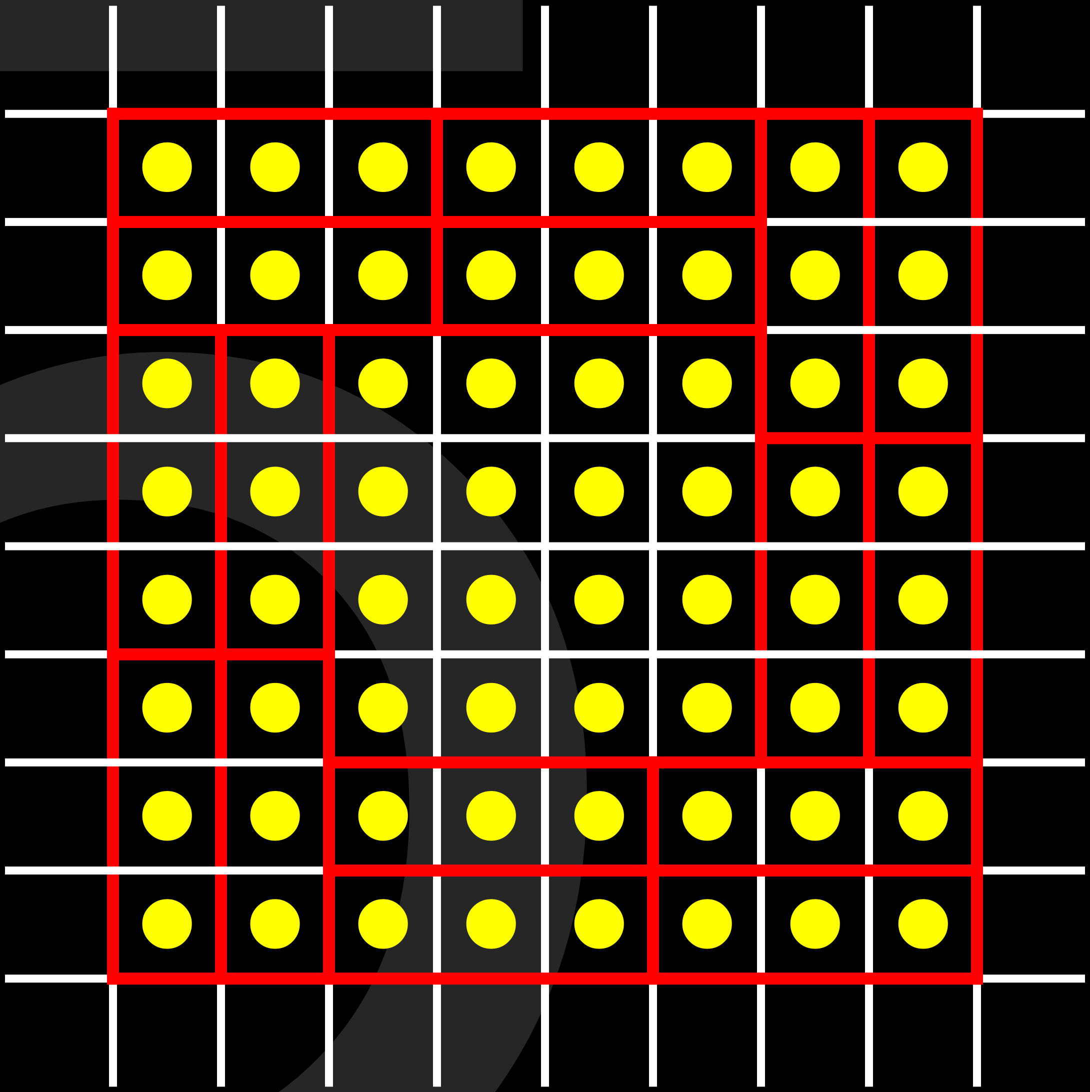
Hint:



$$n = 3(k + 1) + 1$$



$$n = 3(k+1) + 2$$

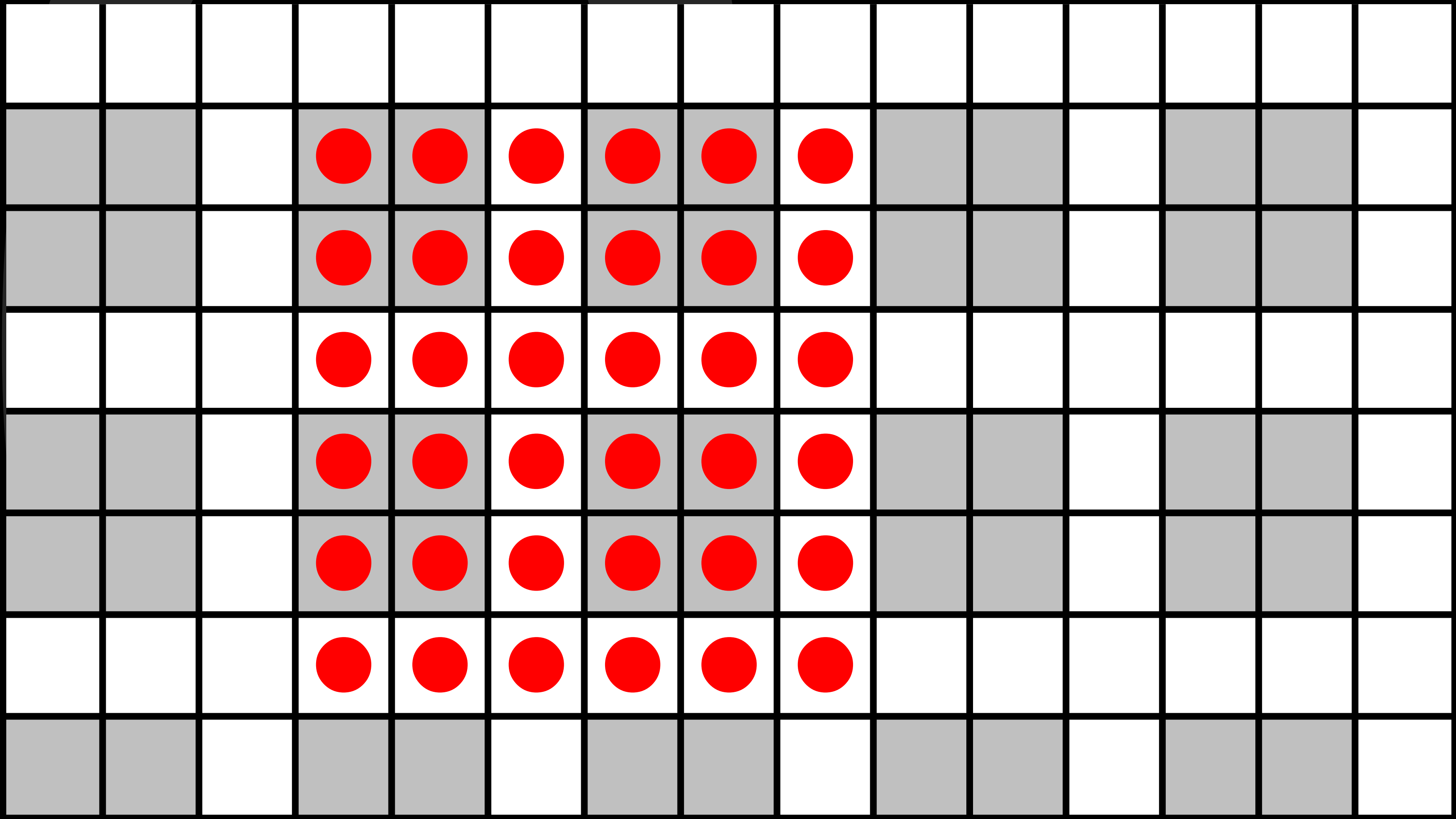


6

Consider the game in the previous question.

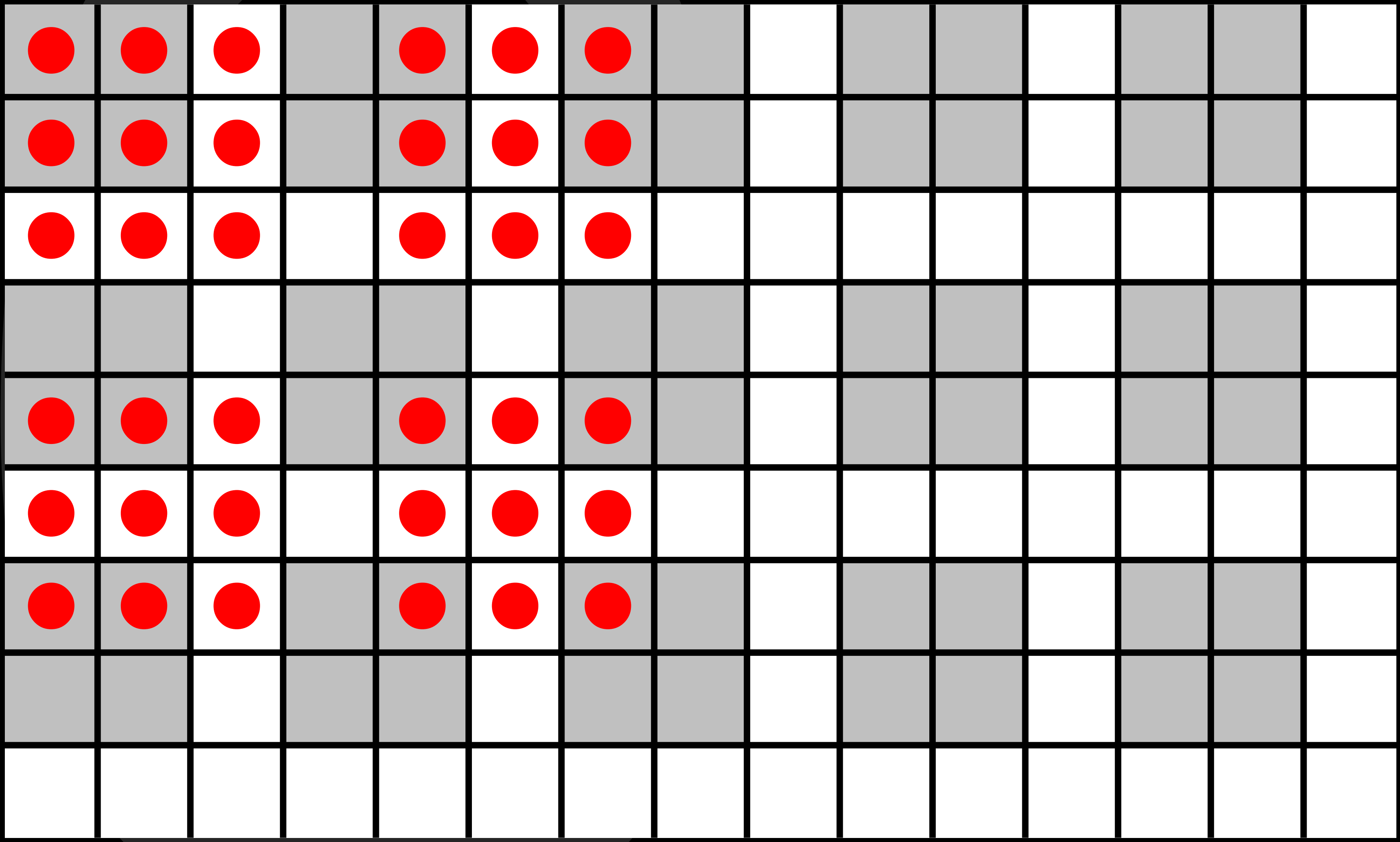
Prove that it is impossible to win the game when *n* is a multiple of 3.

Consider playing the game on such board. By observation, we know that the number of tokens on the black area is always even.



Let’s look at a example

($n = 3$)



Another example:

(n = 6)