CS5319 Advanced Discrete Structure

Homework 4

Due: 3:20 pm, November 24, 2011 (before class)

1. Show that for any k, there is a graph with 2^k vertices such that the graph is k-regular.

2. Fermat once conjectured that for $n \geq 0$, all numbers $F_n = 2^{2^n} + 1$ are primes. Indeed, the numbers
$$F_0 = 3, F_1 = 5, F_2 = 17, F_3 = 257, F_4 = 65537$$
are all primes.

   This conjecture was later disproved by Euler in 1732, who showed that $F_5 = 4294967297 = 641 \times 6700417$. In spite of this, we shall base on it to give an alternative proof (due to Goldbach) that there are infinitely many primes.

   (a) Show that for all $n \geq 1$, $F_n = F_1 \times F_2 \times \cdots \times F_{n-1} + 2$.

   (b) Using the result of (a), argue that Fermat numbers are pairwise relatively prime.

   (c) Show that if we pick one prime factor from each Fermat number, they must be all distinct.

   (d) Using the result of (c), conclude that there are infinitely many primes.

3. Let Δ denote an equilateral triangle with the length of each side equal to 2 units. Show that by placing 5 points inside Δ, we can always find two points whose distance is at most 1 unit. (Hint: Pigeonhole's principle.)

4. Show that among $n + 2$ arbitrarily chosen integers, either there are two whose difference is divisible by $2n$ or there are two whose sum is divisible by $2n$.

5. Consider a game played on an infinite checkerboard where there is an $n \times n$ space and each square in it is occupied by a piece. Each move can jump a piece horizontally or vertically over an adjacent piece on to an empty square, where the jumped-over piece is then removed. The target is to remove the pieces so that there is only one left. Prove that it is possible when n is not a multiple of 3.

6. (Challenge: No Marks) Consider the game in the previous question. Prove that it is impossible to have only one piece left when n is a multiple of 3.