Advanced Discrete Structure Homework 2 Tutorial

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Question 1

Use binomial expansions or combinatorial arguments to evaluate the following sums:

(a) $\sum_{k=0}^{m} \binom{m}{k} \binom{n}{r+k}$

(b) $\sum_{k=0}^{r} (-1)^k \binom{n}{k} \binom{n}{r-k}$

(c) $\sum_{k=0}^{n} 2^k \binom{n}{k}$

Hint

Binomial expansion:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

Two possible way to solve by GF:

1. Find the coefficient of some x^k.
2. Assign x with some value.

Question 2

Find the coefficient of x^{12} in

$$\frac{x+3}{1-2x+x^2}$$

Hint

Two possible ways:

1. Polynomial division.
2. $(1-2x+x^2)^{-1}=?$

Question 3 (a)

Find the ordinary generating function of the sequence $(a_0, a_1, a_2, \ldots)$ where a_r is the number of ways in which the sum r will show when two distinct dice are rolled, with the first one showing even and the second one showing odd.

EX:

$$7 = 2 + 5 = 4 + 3 = 6 + 1$$

Question 3 (b)

Find the ordinary generating function of the sequence $(a_0, a_1, a_2, \dots)$ where a_r is the number of ways in which the sum r will show when 10 distinct dice are rolled, with five of them showing even and the other five showing odd.

EX:

$$35 = 3 + 1 + 3 + 5 + 5$$
$$+ 4 + 6 + 2 + 4 + 2$$

Hint

You can imagine the list of the generating functions of all the possible cases.

Question 4

How many ways are there to collect $24 from 4 children and 6 adults if each person gives at least $1, but each child can give at most $4 and each adult at most $7?

EX:

Children: 2, 2, 3, 4

Adults: 1, 1, 2, 2, 2, 5

Hint

You may need this:

$$\frac{1 - x^{m+1}}{1 - x} = 1 + x + x^2 + \cdots + x^m$$

Question 5

Find the exponential generating function with:

(a) $a_r = 1/(r + 1)$

(b) $a_r = r!$

Question 6

Find the number of n-digit strings generated from the alphabet {0, 1, 2, 3, 4} whose total number of 0's and 1's is even.

EX:

0 2 4 4 3 2 0 1 4 0 3

Hint

Using exponential generating functions!

Two cases:

1. odd 0’s and odd 1’s
2. even 0’s and even 1’s

Question 7 (Challenge)

Show that the number of partitions of n is equal to the number of partitions of $2n$ into n parts.

(Hint: Use Ferrers graph.)

Partition:

Non-distinct objects

Non-distinct groups

Question 8 (Challenge)

Show that for any $s > 1$,

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \equiv \prod_{p\ prime} \frac{1}{1 - p^{-s}}$$

The sum of the left-hand side is popularly known as the Riemann zeta function, $\zeta(s)$, and the overall identity is known as the Euler product formula.