Advanced Discrete Structure Homework 2 Solution

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Question 1

Use binomial expansions or combinatorial arguments to evaluate the following sums:

(a) $\sum_{k=0}^{m} \binom{m}{k} \binom{n}{r+k}$

(b) $\sum_{k=0}^{r} (-1)^k \binom{n}{k} \binom{n}{r-k}$

(c) $\sum_{k=0}^{n} 2^k \binom{n}{k}$

Question 1

(a) $\sum_{k=0}^{m}\binom{m}{k}\binom{n}{r+k}$

$\longrightarrow$ the coefficient of x^r in $(1+x^{-1})^m(1+\mathrm{x})^{\mathrm{n}}$

$$(1+x^{-1})^m(1+\mathrm{x})^{\mathrm{n}} = \left(\frac{1+\mathrm{x}}{\mathrm{x}}\right)^{\mathrm{m}}(1+\mathrm{x})^{\mathrm{n}}$$

$$= \frac{(1+\mathrm{x})^{\mathrm{m+n}}}{\mathrm{x}^{\mathrm{m}}}$$

$\longrightarrow$ the coefficient of x^r: $\binom{m+n}{m+r}$

Question 1

(b) $\sum_{k=0}^{r}(-1)^k\binom{n}{k}\binom{n}{r-k}$

$\rightarrow$ the coefficient of x^r in $(1-x)^n(1+x)^n$

$$(1-x)^n(1+x)^n = (1-x^2)^n$$

$\rightarrow$ the coefficient of x^r:

$$\begin{cases} (-1)^{r/2}\binom{n}{r/2}, & \text{r is even} \\ 0 & \text{r is odd} \end{cases}$$

Question 1

(c) $\sum_{k=0}^{n} 2^k \binom{n}{k}$

$(1+2x)^n$

$$= 1 + 2^1 \binom{n}{1} x + 2^2 \binom{n}{2} x^2 + \cdots + 2^n \binom{n}{n} x^n$$

When $x = 1$:

$$\sum_{k=0}^{n} 2^k \binom{n}{k} = (1+2x)^n = 3^n$$

Question 2

Find the coefficient of x^{12} in

$$\frac{x+3}{1-2x+x^2}$$

Question 2

$$\frac{x+3}{1-2x+x^2} = \frac{x+3}{(1-x)^2}$$
$$= (x+3)(1+x+x^2+\cdots)^2$$

The coefficient of x^{12}:

$$1 \times \binom{11+(2-1)}{2-1} + 3 \times \binom{12+(2-1)}{2-1}$$
$$= 1 \times 12 + 3 \times 13 = 51$$

Question 3 (a)

Find the ordinary generating function of the sequence $(a_0, a_1, a_2, \ldots)$ where a_r is the number of ways in which the sum r will show when two distinct dice are rolled, with the first one showing even and the second one showing odd.

Question 3 (a)

$$(x^2 + x^4 + x^6)(x^1 + x^3 + x^5)$$

Question 3 (b)

Find the ordinary generating function of the sequence $(a_0, a_1, a_2, \dots)$ where a_r is the number of ways in which the sum r will show when 10 distinct dice are rolled, with five of them showing even and the other five showing odd.

Question 3 (b)

E for even, O for odd:

EEEEEOOOOO: $(x^2 + x^4 + x^6)^5 (x^1 + x^3 + x^5)^5$

EOEOEOEOEO: $(x^2 + x^4 + x^6)^5 (x^1 + x^3 + x^5)^5$

EEOOEOEOOE: $(x^2 + x^4 + x^6)^5 (x^1 + x^3 + x^5)^5$

$$\vdots$$

The sum: $\frac{10!}{5!5!}(x^2 + x^4 + x^6)^5 (x^1 + x^3 + x^5)^5$

Question 4

How many ways are there to collect \$24 from 4 children and 6 adults if each person gives at least \$1, but each child can give at most \$4 and each adult at most \$7?

Question 4

$$(x + x^2 + \cdots + x^4)^4 (x + x^2 + \cdots + x^7)^6$$
$$= \mathrm{x}^{10}(1 + x + x^2 + x^3)^4 (1 + x + x^2 + \cdots + x^6)^6$$
$$= \mathrm{x}^{10}\left(\frac{1 - \mathrm{x}^4}{1 - \mathrm{x}}\right)^4 \left(\frac{1 - \mathrm{x}^7}{1 - \mathrm{x}}\right)^6 = \frac{x^{10}(1 - x^4)^4 (1 - x^7)^6}{(1 - x)^{10}}$$

Question 4

$$(1-x^4)^4 = 1 - 4x^4 + 6x^8 - 4x^{12} + x^{16}$$
$$(1-x^7)^6 = 1 - 6x^7 + 15x^{14} - 20x^{21} + 15x^{28} - 6x^{35} + x^{42}$$
$$1 \times 1 \times \binom{14+9}{9} + 1 \times (-6) \times \binom{7+9}{9}$$
$$+1 \times 15 \times \binom{0+9}{9} + (-4) \times 1 \times \binom{10+9}{9}$$
$$+(-4) \times (-6) \times \binom{3+9}{9} + 6 \times 1 \times \binom{6+9}{9}$$
$$+(-4) \times 1 \times \binom{2+9}{9}$$

Question 5

Find the exponential generating function with:

(a) $a_r = 1/(r + 1)$

(b) $a_r = r!$

Question 5

(a) $a_r = \dfrac{1}{r+1}$

$$1 + \left(\frac{1}{2}\right)\frac{x}{1!} + \left(\frac{1}{3}\right)\frac{x}{2!} + \cdots = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots$$

$$= \frac{\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) - 1}{x} = \frac{e^x - 1}{x}$$

Question 5

(b) $a_r = r!$

$$1 + 1!\left(\frac{x}{1!}\right) + 2!\left(\frac{x^2}{2!}\right) + \cdots = 1 + x + x^2 + \cdots$$

$$= \frac{1}{1-x}$$

Question 6

Find the number of n-digit strings generated from the alphabet {0, 1, 2, 3, 4} whose total number of 0's and 1's is even.

Question 6

The number of 0's and 1's are both even:

$$\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)^2\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)^3$$

$$=\left(\frac{e^x+e^{-x}}{2}\right)^2(\mathrm{e}^{\mathrm{x}})^3=\frac{\mathrm{e}^{5\mathrm{x}}+\mathrm{e}^{3\mathrm{x}}+\mathrm{e}^{\mathrm{x}}}{4}$$

Question 6

The number of 0's and 1's are both odd:

$$\left(x+\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots\right)^2\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)^3$$

$$=\left(\frac{e^x-e^{-x}}{2}\right)^2(\mathrm{e}^{\mathrm{x}})^3=\frac{\mathrm{e}^{5\mathrm{x}}-\mathrm{e}^{3\mathrm{x}}+\mathrm{e}^{\mathrm{x}}}{4}$$

Question 6

The sum:

$$\frac{e^{5x} - e^{3x} + e^{x}}{4} + \frac{e^{5x} - e^{3x} + e^{x}}{4} = \frac{e^{5x} + e^{x}}{2}$$

The coefficient of $\frac{x^n}{n!}$ is:

$$\frac{5^n + 1}{2}$$

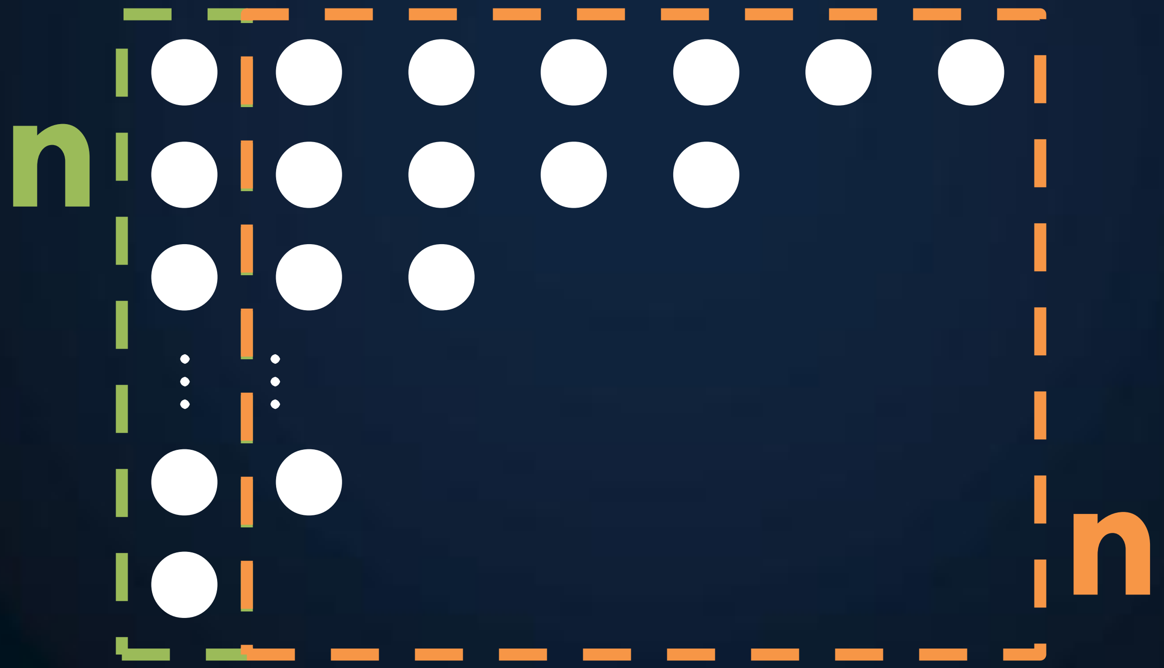
Question 7 (Challenge)

Show that the number of partitions of n is equal to the number of partitions of $2n$ into n parts.

(Hint: Use Ferrers graph.)

Question 7 (Challenge)

To divide 2n objects into n groups:



A Interesting Fact about Ferr's Graph

The number of partitions with no one larger than k

=

The number of partitions with no more than k groups

Question 8 (Challenge)

Show that for any $s > 1$,

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \equiv \prod_{p\ prime} \frac{1}{1 - p^{-s}}$$

The sum of the left-hand side is popularly known as the Riemann zeta function, $\zeta(s)$, and the overall identity is known as the Euler product formula.

Question 8 (Challenge)

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots$$

$$\frac{1}{2^s}\zeta(s) = \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \cdots$$

$$\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \cdots$$

Question 8 (Challenge)

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Question 8 (Challenge)

$$\left(1-\frac{1}{2^s}\right)\zeta(s)=1+\frac{1}{3^s}+\frac{1}{5^s}+\frac{1}{7^s}+\cdots$$

$$\frac{1}{3^s}\left(1-\frac{1}{2^s}\right)\zeta(s)=\frac{1}{3^s}+\frac{1}{9^s}+\frac{1}{15^s}+\frac{1}{21^s}+\cdots$$

$$\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{2^s}\right)\zeta(s)=\frac{1}{5^s}+\frac{1}{7^s}+\frac{1}{11^s}+\frac{1}{13^s}+\cdots$$

Question 8 (Challenge)

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Question 8 (Challenge)

Repeating this:

$$\cdots\left(1-\frac{1}{11^s}\right)\left(1-\frac{1}{7^s}\right)\left(1-\frac{1}{5^s}\right)\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{2^s}\right)\zeta(s)=1$$

And we can get:

$$\zeta(s)=\frac{1}{\left(1-\frac{1}{2^s}\right)\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{5^s}\right)\left(1-\frac{1}{7^s}\right)\left(1-\frac{1}{11^s}\right)\cdots}$$

$$=\prod_{p\ prime}\frac{1}{1-p^{-s}}$$