Advanced Discrete Structure Homework 1 Solution

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Question 1

Suppose a coin is tossed 12 times and there are 3 heads and 9 tails.

How many such sequences are there in which there are at least 5 tails in a row?

Question 1

We use I for head and O for tail:

OIOOOOOIOOIO

Clearly, the tails were divided into 4 groups by the heads.

Question 1

Also, 5 is just larger than half of 9.

Therefore, we can try to find the number of ways to divide 9 tails into 4 distinct groups with one group have at least 5 members.

$$\mathbf{4} \times \binom{\mathbf{4} + (\mathbf{4} - \mathbf{1})}{\mathbf{4} - \mathbf{1}} = \mathbf{140}$$

Question 2

How many non-negative integer solutions are there to the equation

$$\mathbf{2x_1 + 2x_2 + x_3 + x_4 = 12?}$$

Question 2

It can be divided into 7 cases:

When $2x_1 + 2x_2 = 0$ $\qquad \binom{0 + (2-1)}{2-1} \times \binom{12 + (2-1)}{2-1}$

When $2x_1 + 2x_2 = 2$ $\qquad \binom{1 + (2-1)}{2-1} \times \binom{10 + (2-1)}{2-1}$

When $2x_1 + 2x_2 = 4$ $\qquad \binom{2 + (2-1)}{2-1} \times \binom{8 + (2-1)}{2-1}$

When $2x_1 + 2x_2 = 6$ $\qquad \binom{3 + (2-1)}{2-1} \times \binom{6 + (2-1)}{2-1}$

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Question 2

The sum: 140

Question 3 (a)

Show that the total number of permutations of p red balls and 0, or 1, or 2, …, or q white balls is

$$\frac{p!}{p!} + \frac{(p+1)!}{p!\,1!} + \frac{(p+2)!}{p!\,2!} + \cdots + \frac{(p+q)!}{p!\,q!}$$

Question 3 (a)

p red balls and x white balls:

$$\frac{(\boldsymbol{p}+\boldsymbol{x})!}{\boldsymbol{p}!\,\boldsymbol{x}!}$$

The total:

$$\frac{\boldsymbol{p}!}{\boldsymbol{p}!}+\frac{(\boldsymbol{p}+\mathbf{1})!}{\boldsymbol{p}!\,\mathbf{1}!}+\frac{(\boldsymbol{p}+\mathbf{2})!}{\boldsymbol{p}!\,\mathbf{2}!}+\cdots+\frac{(\boldsymbol{p}+\boldsymbol{q})!}{\boldsymbol{p}!\,\boldsymbol{q}!}$$

Question 3 (b)

Show that the sum in part (a) is

$$\frac{(\boldsymbol{p}+\boldsymbol{q}+\mathbf{1})!}{(\boldsymbol{p}+\mathbf{1})!\,\boldsymbol{q}!}$$

Question 3 (b)

$$\frac{(\boldsymbol{p}+\boldsymbol{q}+\mathbf{1})!}{(\boldsymbol{p}+\mathbf{1})!\,\boldsymbol{q}!}$$

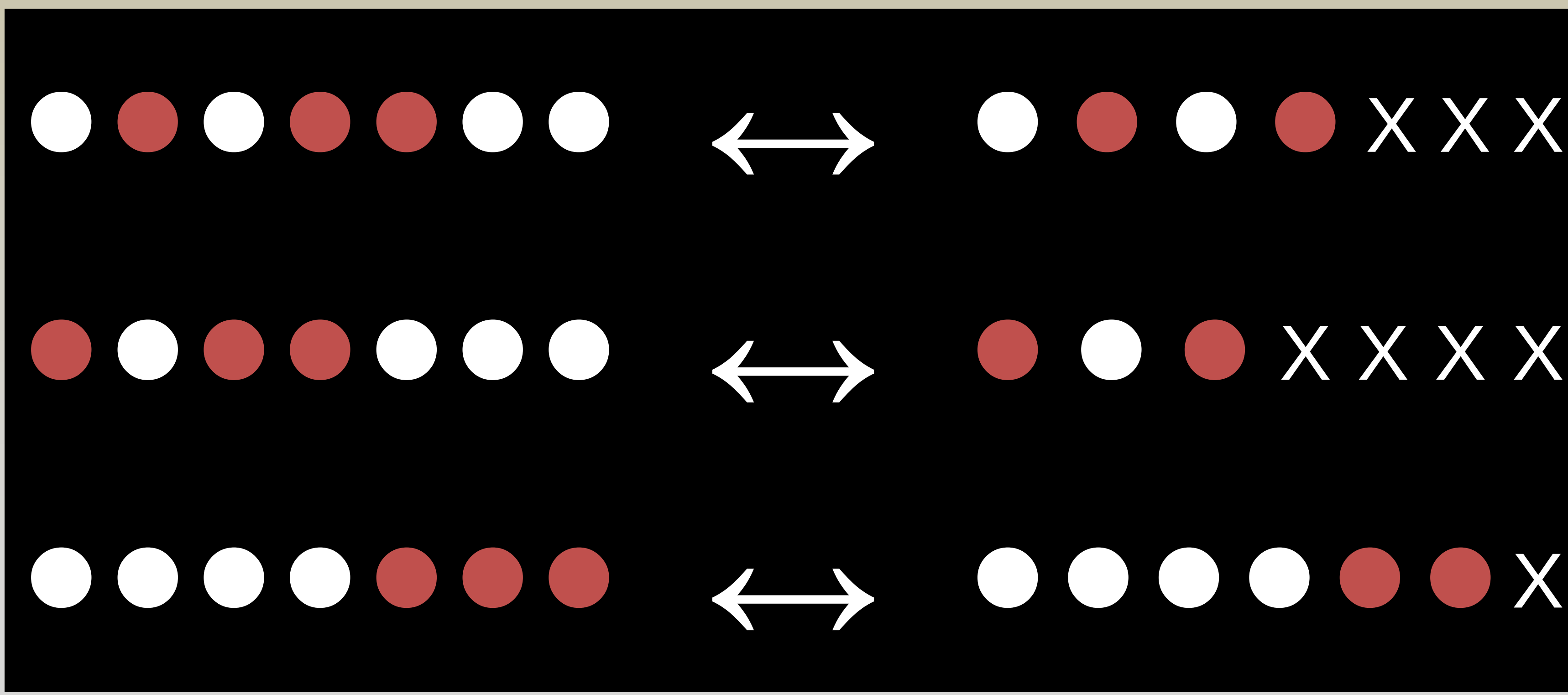
Suppose that we are arranging

$(p + 1)$ red balls and q white balls.

Question 3 (b)

p + 1 = 3, q = 4

Delete the last red ball and the balls follow it:



Question 3 (c)

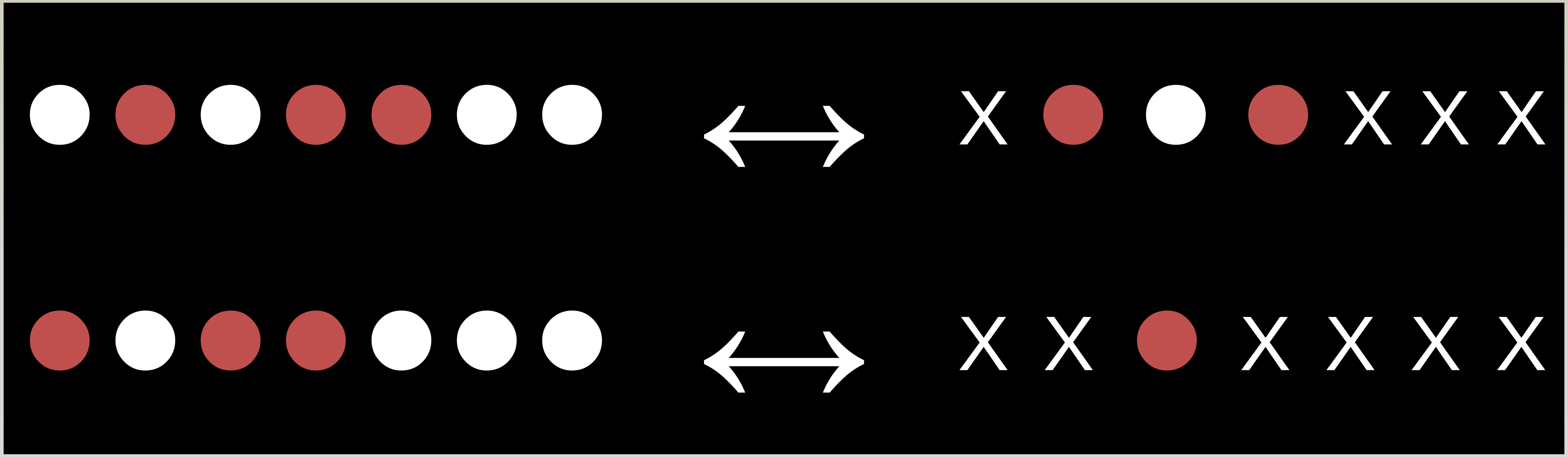
Show that the total number of permutations of 0, or 1, or 2, …, or p red balls with 0, or 1, or 2, …, or q white balls is

$$\frac{(p+q+2)!}{(p+1)!\,(q+1)!} - 1$$

Question 3 (c)

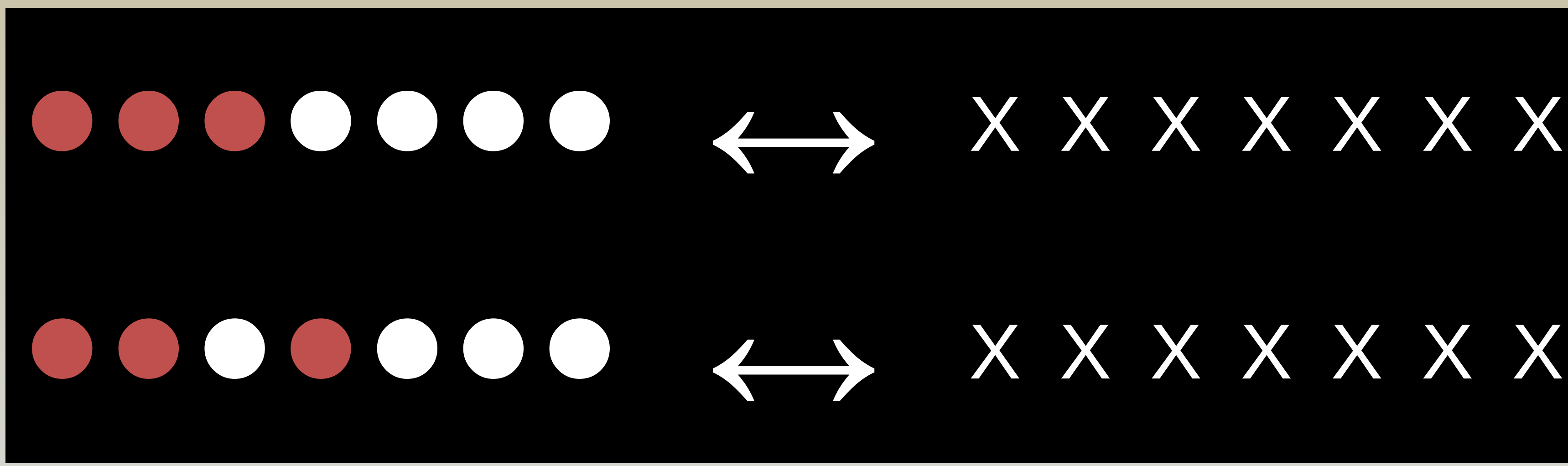
$p + 1 = 3,\ q + 1 = 4$

Delete the last red ball and the balls after it, and then delete the first white ball and the balls before it:



Question 3 (c)

Over counted cases:



Question 4

How many arrangements are there of seven *a*s, eight *b*s, three *c*s, and six *d*s with no occurrence of the consecutive pairs *ca* or *cc*?

Question 4

Arrange *a*s, *b*s, and *d*s first:

_a_b_b_a_a_b_a_b_b_a_b_d_d_d_a_d_d_b_b_a_d_

We have 22 places to insert *c*.

Question 4

To avoid *cc*:

Select 3 distinct places.

To avoid *ca*:

Select places not before *a*.

Question 4

Therefore, the total number of permutation is

$$\frac{\mathbf{21}!}{\mathbf{7}!\,\mathbf{8}!\,\mathbf{6}!} \times \boldsymbol{C}(\mathbf{22}-\mathbf{7},\mathbf{3}) = \mathbf{158880922200}$$

Question 5

How many ways are there to distribute 25 dffierent presents to four people (including the boss) at an office party so that the boss receives exactly twice as many presents as the second popular person?

Question 5

4 cases:

The second popular person got 5 presents:

$$\left(\binom{25}{10}\times\binom{15}{5}\times\binom{10}{5}\right)$$

The second popular person got 6 presents:

$$\left(\binom{25}{12}\times\binom{13}{6}\times\left(\sum_{i=1}^{6}\binom{7}{i}-\binom{7}{6}/2-\binom{7}{1}/2\right)\right)\times 3$$

The second popular person got 7 presents:

$$\left(\binom{25}{14}\times\binom{11}{7}\times\left(\left(\sum_{i=0}^{4}\binom{4}{i}\right)-\binom{4}{2}/2\right)\right)\times 3$$

The second popular person got 8 presents:

$$\left(\binom{25}{16}\times\binom{11}{8}\times\left(\left(\sum_{i=0}^{2}\binom{2}{i}\right)-\binom{2}{1}/2\right)\right)\times 3$$

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Question 6

Professor Grinch's telephone number is 6328363. Mickey remembers the collections of digits but not their order, except that he knows the first 6 is before the first 3. How many arrangements of these digits with this constraint are there?

Question 6

Replace 6s and 3s with Xs:

XXX8XX2

Replace the first X with 6

6XX8XX2

The first 6 will be always before the first 3.

Question 6

XXX8XX2

$\frac{\mathbf{7!}}{\mathbf{1!1!5!}}$ ways of arrangement

6XX8XX2

1 way

6338632

$\frac{\mathbf{4!}}{\mathbf{1!3!}}$ ways of arrangement

Question 6

$$\frac{7!}{1!\,1!\,5!} \times 1 \times \frac{4!}{1!\,3!} = 168$$

Question 7

A man has seven friends. How many ways are there to invite a different subset of three of these friends for a dinner on seven successive nights such that each pair of friends are together at just one dinner?

Question 7

Each friend will meet 2 others at one dinner.

→At most 3 dinners for each friend.

7 days and 3 friend per day.

→21 in total

Each friend will be invited exactly 3 times.

Question 7

Suppose that the friends are A, B, C, …, G.

Ignore the arrangement of days first.

Consider the days when A is invited:

$$\left(\binom{\mathbf{6}}{\mathbf{2}} \times \binom{\mathbf{4}}{\mathbf{2}} \times \binom{\mathbf{2}}{\mathbf{2}}\right)/\mathbf{3}! = \mathbf{15}$$

Question 7

Suppose In B's view, the three days are:

(A, B, v), (A, w, x), (A, y, z)

The friends correspond to w and y are before those correspond to x and z in alphabet order.

For the other two days of B:

B, w, y	B, w, z
B, x, z	B, x, y

Only one choice for v.

Question 7

The answer is:

$$\mathbf{15 \times 2 \times 7! = 151200}$$