CS5319 Advanced Discrete Structure

Homework 6

Due: 1:10 pm, January 3, 2011 (before class)

1. Let $G = \{q \mid q \in \mathbb{Q} \text{ and } q \neq -1\}$. Define $\otimes$ be a binary operator such that

$$x \otimes y = x + y + xy.$$

   Show that $(G, \otimes)$ is a group.

2. Suppose $(G, \star)$ is a group, and both $(H, \star)$ and $(K, \star)$ are subgroups of $(G, \star)$.
   Show that $(H \cap K, \star)$ is a subgroup of $(G, \star)$.

3. Let p be a prime. Let $(G, \star)$ be a group.
   (a) If the order of G is $2p$, show that every proper subgroup of $(G, \star)$ is cyclic.
   (b) If the order of G is p^2, show that there exists a subgroup of $(G, \star)$ whose order is p.

4. A rod is divided into six segments, and each segment will be colored by one of the n colors. In how many distinct ways can the rod be colored? (For this question, we assume that two colorings are equal if one can be transformed to the other by rotating the rod by 180°.)

5. Design a finite automaton that accepts exactly all binary strings each of which ends with `11`. For instance, the automaton should accept `11`, `0111`, but should reject `1`, `0110`, or the empty string.

6. A *palindrome* is a string that reads the same forward and backward. For instance, `010` and `1001` are palindromes, but `1010` is not. Show that the language

$$L = \{w \mid \ w \text{ is a binary string and } w \text{ is a palindrome}\}$$

   is non-regular.

7. Define `HAMPATH` and `HAMCIRCUIT` to be the problems as follows.
   - `HAMPATH`: Given an input graph G, is there a Hamiltonian path in G?
   - `HAMCIRCUIT`: Given an input graph G, is there a Hamiltonian circuit in G?

   In this question, we want to show that these two problems are equivalent under polynomial-time reduction.
   (a) Show that we can reduce `HAMPATH` to `HAMCIRCUIT` in polynomial time.
   (b) Show that we can reduce `HAMCIRCUIT` to `HAMPATH` in polynomial time.

8. It is known that `HAMPATH` is NP-complete. Show that the following problem, called `LONGEST-PATH`, is also NP-complete.
   - Given an input graph G and an integer K, does G contain a simple path whose length is at least K?