1. For each \( m \) greater than 1, how many primes are there in the closed interval \([m!+2, m!+m]\)? Explain your answer.

2. Let \( p \) be a prime.
   (a) Suppose \( xy \equiv 0 \pmod{p} \). Show that either \( x \equiv 0 \) or \( y \equiv 0 \) modulo \( p \).
   (b) Show that if \( 1 < x < p - 1 \) and \( xy \equiv 1 \pmod{p} \), then \( y \not\equiv x \pmod{p} \).
   (c) Show that \( 2 \times 3 \times \cdots \times (p - 2) \equiv 1 \pmod{p} \).
   (d) Conclude that Wilson’s theorem is true. That is, \((p - 1)! \equiv -1 \pmod{p}\).

3. Prove that if \( p \) is a prime congruent to 1 modulo 4, then \( \left(\frac{p - 1}{2}\right)!^2 \equiv -1 \pmod{p} \).
   \textbf{Hint:} Show that \( ((p - 1)/2)!^2 \equiv (p - 1)! \pmod{p} \).

4. Prove that if \( n^j \equiv 1 \pmod{m} \) and \( n^k \equiv 1 \pmod{m} \), then \( n^{\gcd(j,k)} \equiv 1 \pmod{m} \).
   \textbf{Hint: Properties of GCD.}

5. A number \( n \) is a perfect number if the sum of all the proper divisors of \( n \) (i.e., all divisors excluding \( n \) itself) is exactly \( n \). For instance, 6 and 28 are both perfect numbers, because
   \[
   \text{sum of proper divisors of 6} = 1 + 2 + 3 = 6, \quad \text{and} \quad \text{sum of proper divisors of 28} = 1 + 2 + 4 + 7 + 14 = 28.
   \]
   In the following, we shall show an interesting result by Euler:

\textbf{Theorem 1.} An even number \( n \) is a perfect number if and only if \( n = 2^m(2^{m+1} - 1) \) and \( 2^{m+1} - 1 \) is prime.

(a) Prove that if \( n = 2^m(2^{m+1} - 1) \) and \( 2^{m+1} - 1 \) is a prime, then \( n \) is a perfect number.
(b) Suppose \( n \) is an even number, so that we can express \( n \) as \( 2^mQ \) for some odd integer \( Q \). Also, suppose \( \sigma(Q) \) denotes the sum of all divisors of \( Q \) (i.e., including itself). Show that if \( n \) is a perfect number, then
   \[
   2^{m+1}Q = 2n = (2^{m+1} - 1)\sigma(Q).
   \]
   (c) Using the result from part (b), show that \( Q \) is a multiple of \( 2^{m+1} - 1 \).
   (d) Suppose that \( Q = (2^{m+1} - 1)q \). Show that the following is true:
   \[
   2^{m+1}q = \sigma(Q) \geq q + Q = 2^{m+1}q.
   \]
   (e) Using the result from part (d), show that \( Q \) must be a prime and \( Q = 2^{m+1} - 1 \). In other words, \( n = 2^mQ = 2^m(2^{m+1} - 1) \) for some prime \( Q = 2^{m+1} - 1 \).

6. (Challenging: No marks) Show that for all \( n > 1 \), \( 2^n \not\equiv 1 \pmod{n} \).