CS5319 Advanced Discrete Structure

Homework 2

Due: 1:10 pm, October 21, 2010 (before class)

1. Suppose m and n are non-negative, and $k \leq \min(m, n)$. What is the following sum? (Explain why your sum is correct.)
$$\binom{n}{0}\binom{m}{k} + \binom{n}{1}\binom{m}{k-1} + \cdots + \binom{n}{k}\binom{m}{0}$$

2. Find the value of a_{50} in the following expansion:
$$\frac{x-3}{x^2-3x+2} = a_0 + a_1x + a_2x^2 + \cdots + a_{50}x^{50} + \cdots$$

3. Show that
$$\frac{1}{1-x} \equiv (1+x+x^2)(1+x^3+x^6)(1+x^9+x^{18})\cdots$$
In terms of partition of integers, what is so special about the above identity?

4. In how many ways can $3n$ letters be selected from $2n$ A's, $2n$ B's, and $2n$ C's? (Order of letters is not important.)

5. Find the number of n-digit strings generated from the alphabet $\{0, 1, 2, 3, 4\}$ whose number of 0's and number of 1's are both even.

6. Find the number of n-digit strings generated from the alphabet $\{0, 1, 2, 3, 4\}$ whose *total number* of 0's and 1's is even.

7. Find the exponential generating function of the sequence:
$$(1, 1 \times 4, 1 \times 4 \times 7, \ldots, 1 \times 4 \times \cdots \times (3r+1), \ldots)$$

8. (Challenging: No marks) Suppose n and k are non-negative, with $k \leq n$. What is the following sum? (Explain why your sum is correct.)
$$\binom{n}{0}\binom{n}{k} - \binom{n}{1}\binom{n-1}{k-1} + \cdots + (-1)^k\binom{n}{k}\binom{n-k}{0}$$

9. (Challenging: No marks)
   (a) Evaluate the definite integral
   $$\int_0^\infty e^{-s}s^k\,ds.$$
   (b) Let $A(x)$ and $E(x)$ be the ordinary and exponential generating functions of the sequence of numbers $(a_0, a_1, a_2, \ldots)$, respectively. Show that
   $$A(x) = \int_0^\infty e^{-s}E(sx)\,ds.$$

10. (Challenging: No marks) Let $p_3(n)$ denote the number of ways to partition the integer n into at most 3 parts. Show that the number of different triangles with integral sides and perimeter n is equal to

$$p_3\left(\frac{n-3}{2}\right) \text{ when } n \text{ is odd, and } p_3\left(\frac{n-6}{2}\right) \text{ when } n \text{ is even}$$

Hint: Suppose a, b, c are the lengths of the three sides of a triangle, with $a \geq b \geq c$. Then we must have:

$$a + b - c \geq a + c - b \geq b + c - a > 0.$$

How about the converse?